Settling the Threshold Degree and Sign Rank of $\mathbf{AC}^0$

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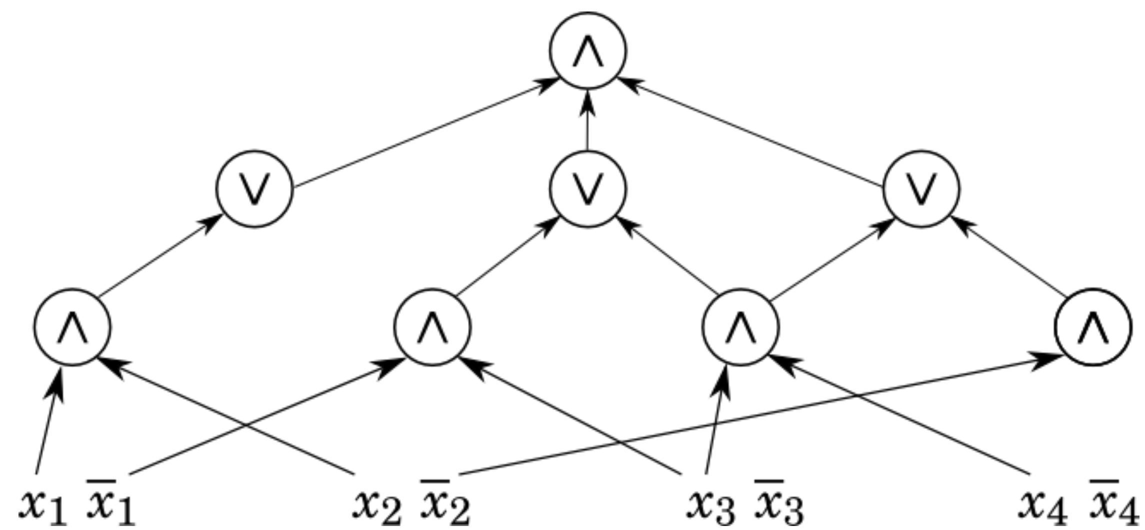
UCLA

Constant depth circuits (AC^0)



constant depth, polynomial #gates ($\wedge$, $\vee$, $\neg$)

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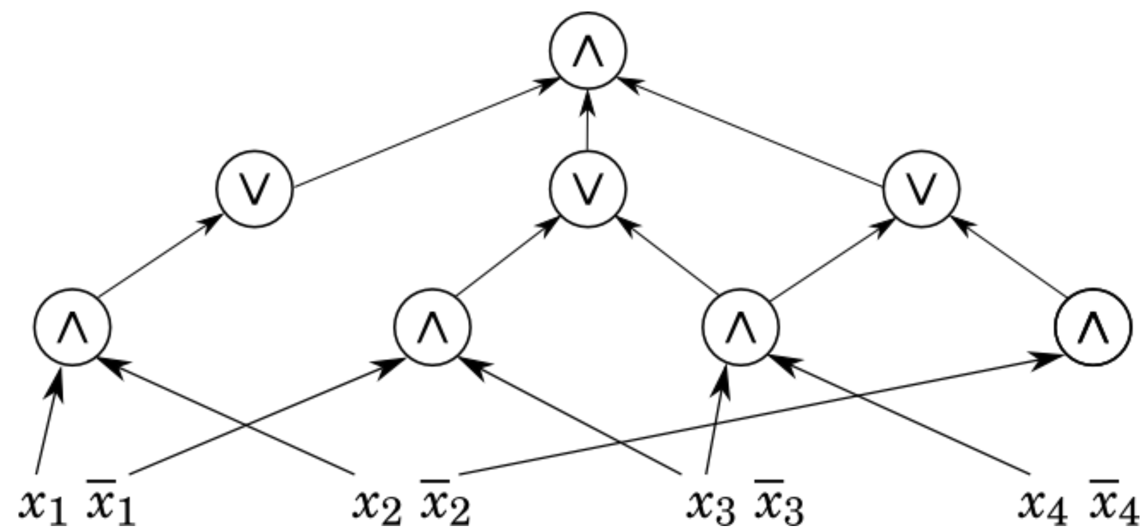
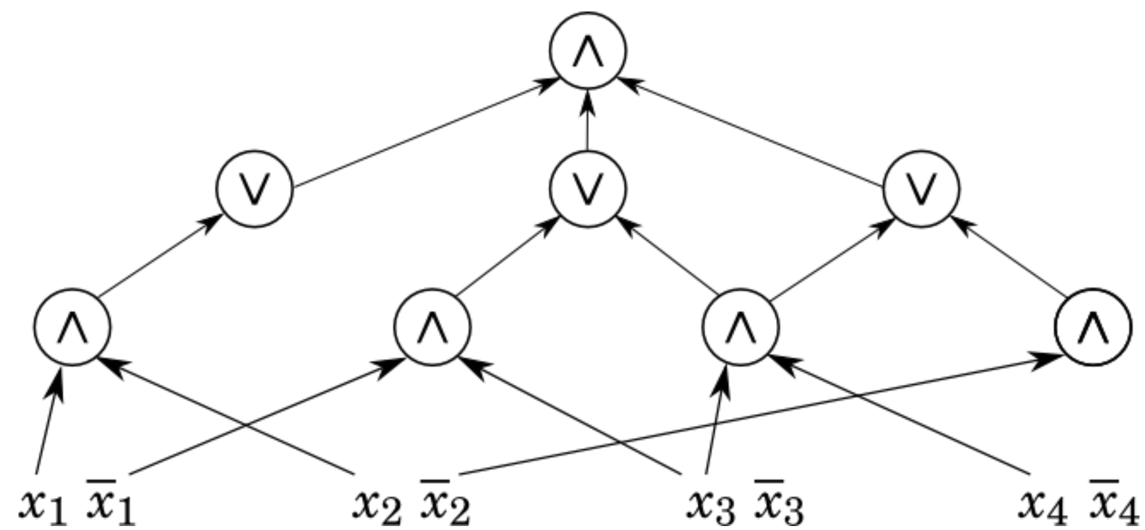


constant depth, polynomial #gates ($\wedge$, $\vee$, $\neg$)

Why study AC^0?

Simple, natural computational model,
has some of the most impressive results

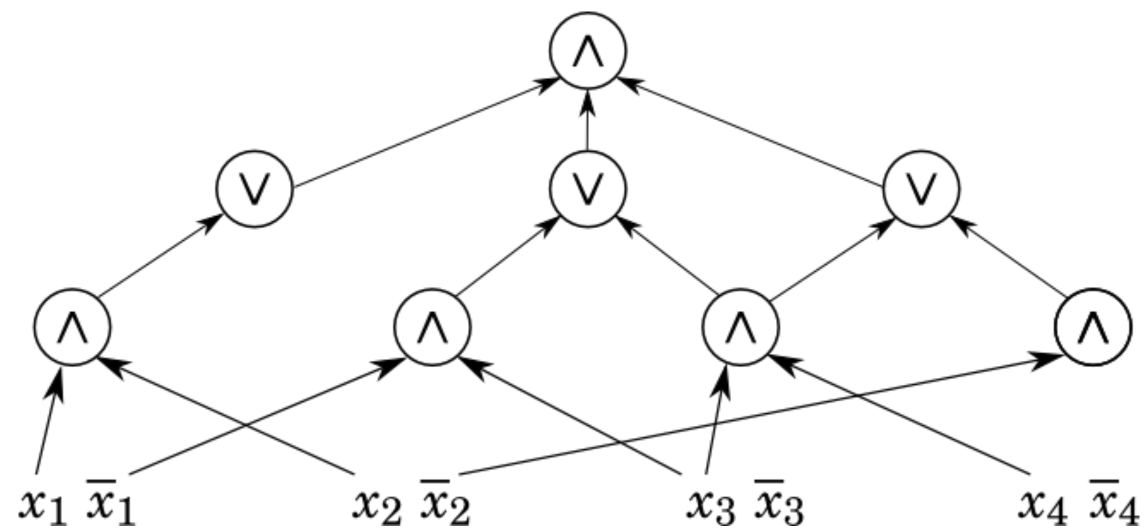
Constant depth circuits (AC^0)



Circuits
lower bound
"P vs NP"

[FSS84, Ajt83, Yao85, Has86, Aar10, RS10, LV11, BIL12, IMP12, Has14, AA15, LRR17, Ros18, Vio18]

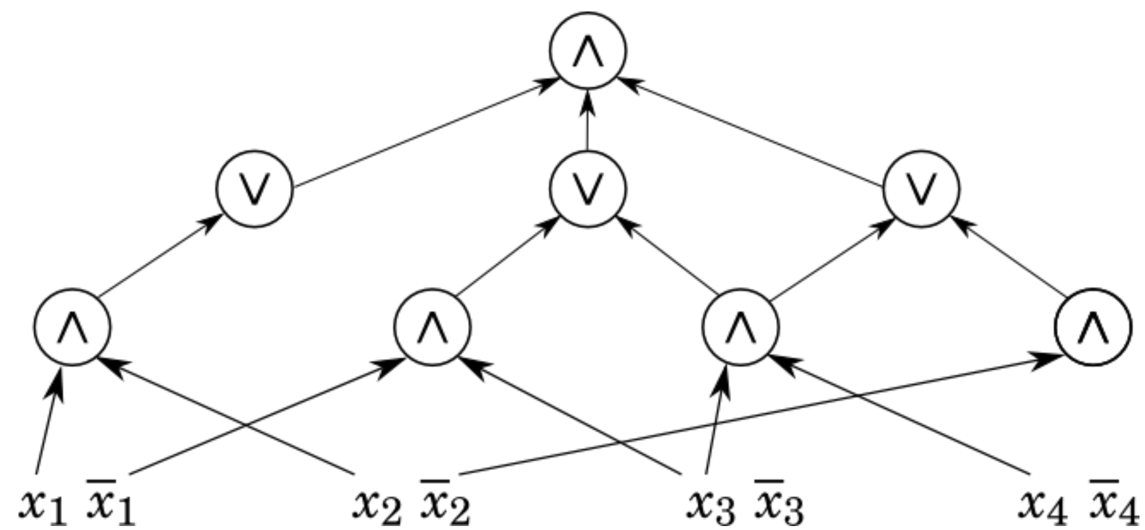
Constant depth circuits (AC^0)



Communication complexity

[AFR85, PS86, KS92, Raz92, FKLMS01, F02, CA08, RS08, S09, BH12, S14]

Constant depth circuits (AC^0)



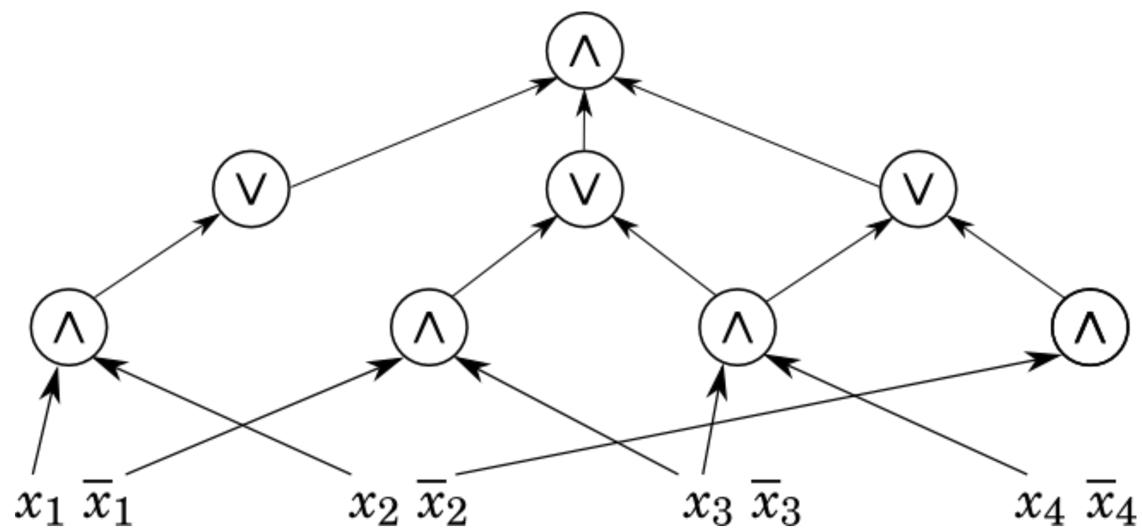
Communication complexity

[AFR85, PS86, KS92, Raz92, FKLMS01, F02, CA08, RS08, S09, BH12, S14]

"P vs BPP"

[LN90, Nis91, Baz07, Raz08, Bra09, ETT10, GMR13, TX13, Tal14, CSV15, HS16, Tal17, ST18, DHH18,]

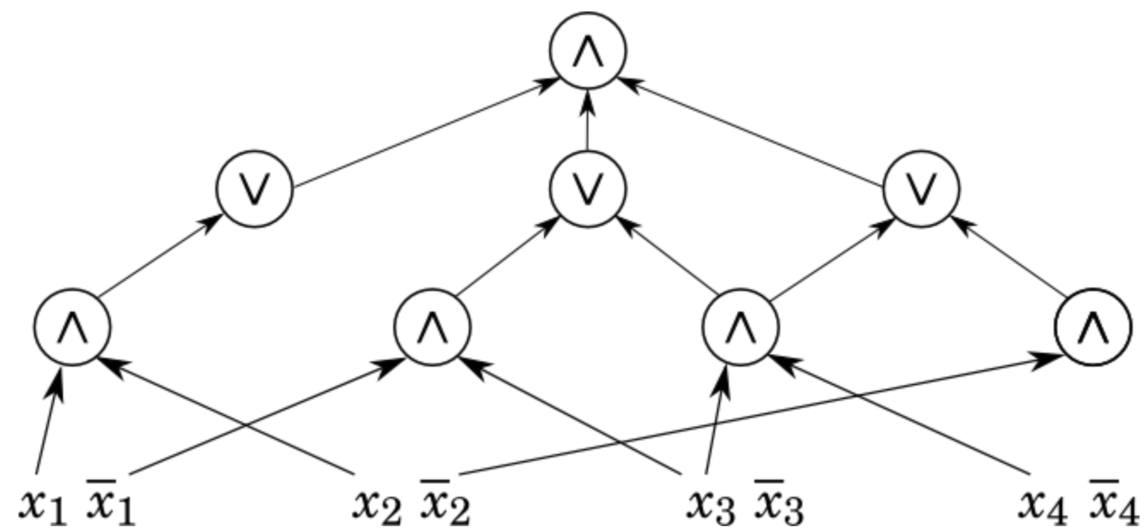
Constant depth circuits (AC^0)



Quantum supremacy?

[AS04, Amb07, ACR+10, BM10, Rei10, Bel12, BS13, RT19]

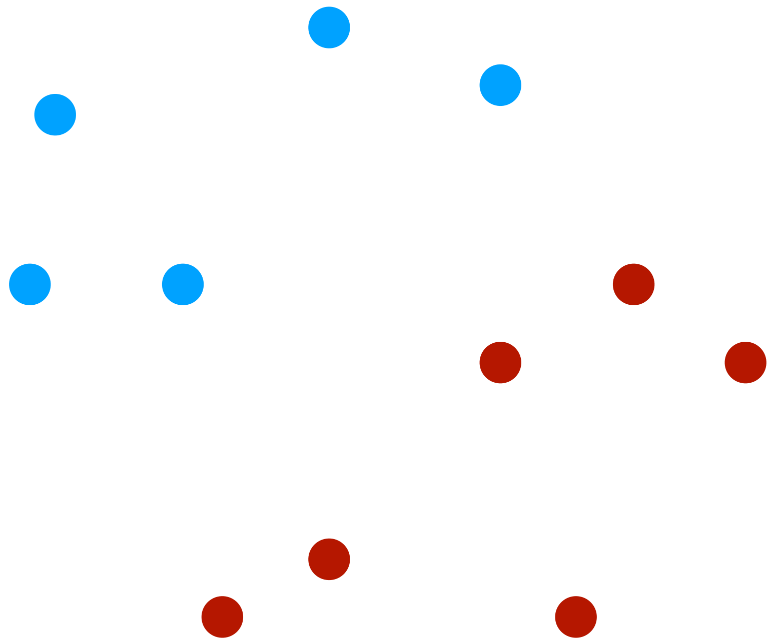
Constant depth circuits (AC^0)



Quantum supremacy?	[AS04, Amb07, ACR+10, BM10, Rei10, Bel12, BS13, RT19]
Learning	[LMN93, Jac02, BES03, OS03, KOS04, KS04, LMSS07, AMY16, DRG17]

……

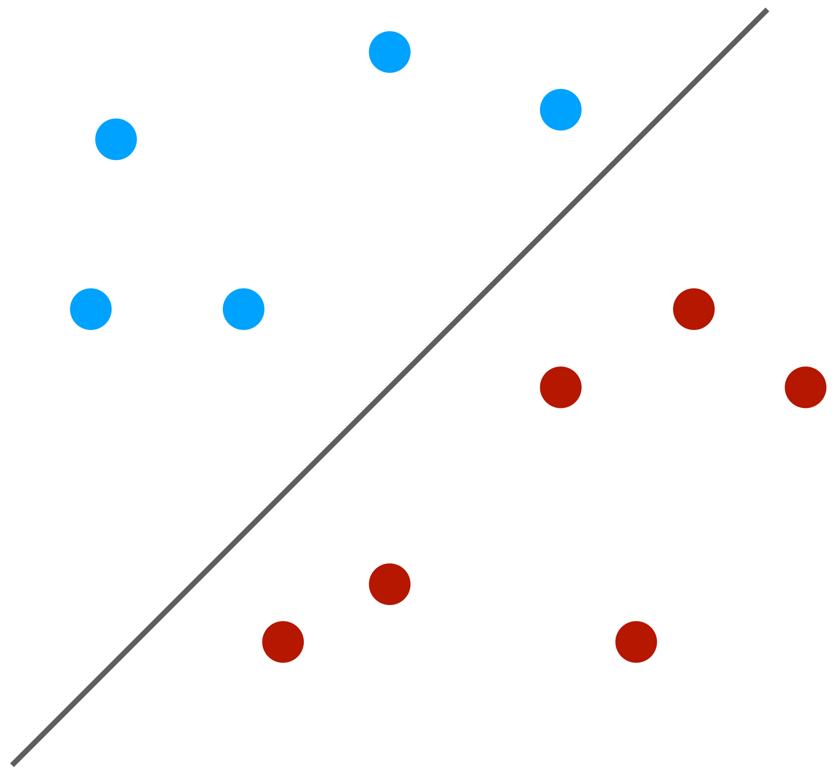
Why study sign representations? Learning halfspace



$$a_1x_1 + a_2x_2 + a_3x_3 + a_0 \geq 0$$

learn the coefficients a

Why study sign representations?
Learning halfspace



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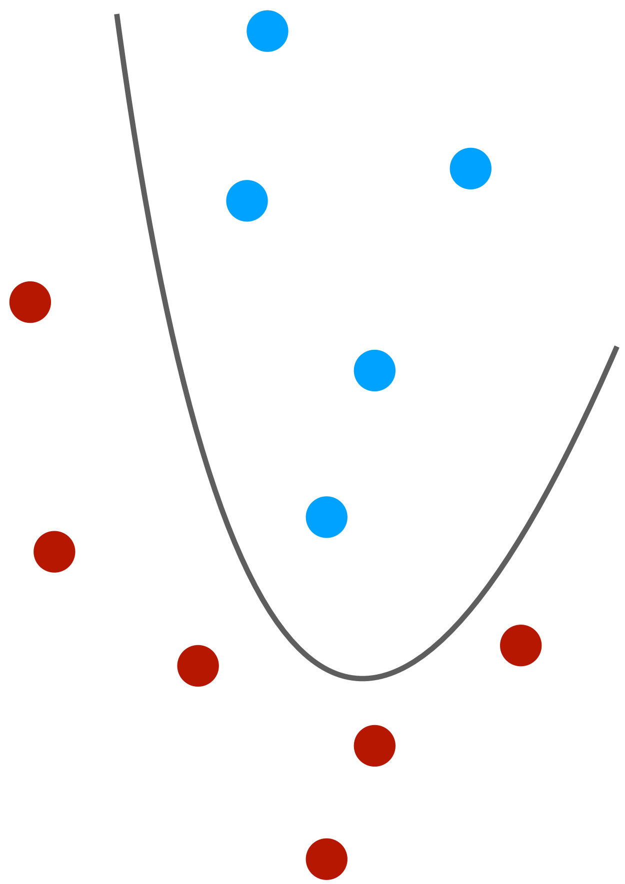
Learning low degree polynomials



$$a_1x_1 + a_2x_2 + a_3x_3 + a_{12} \cdot x_1x_2 + a_{13} \cdot x_1x_3 + a_{23} \cdot x_2x_3 \geq 0$$

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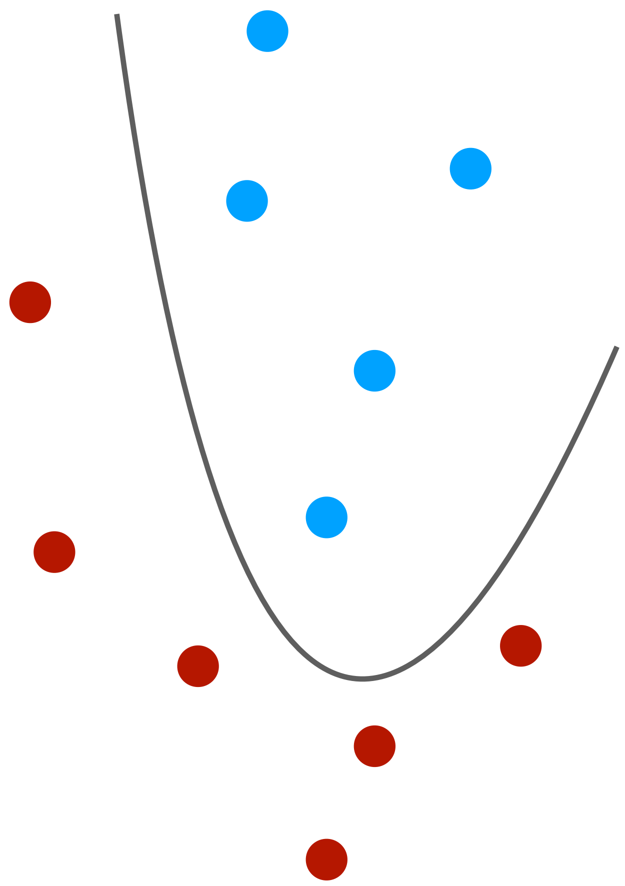

$$a_1x_1 + a_2x_2 + a_3x_3 + a_{12} \cdot \overset{y_{12}}{\cancel{x_1x_2}} + a_{13} \cdot \underset{y_{13}}{\cancel{x_1x_3}} + a_{23} \cdot \underset{y_{23}}{\cancel{x_2x_3}} \geq 0$$

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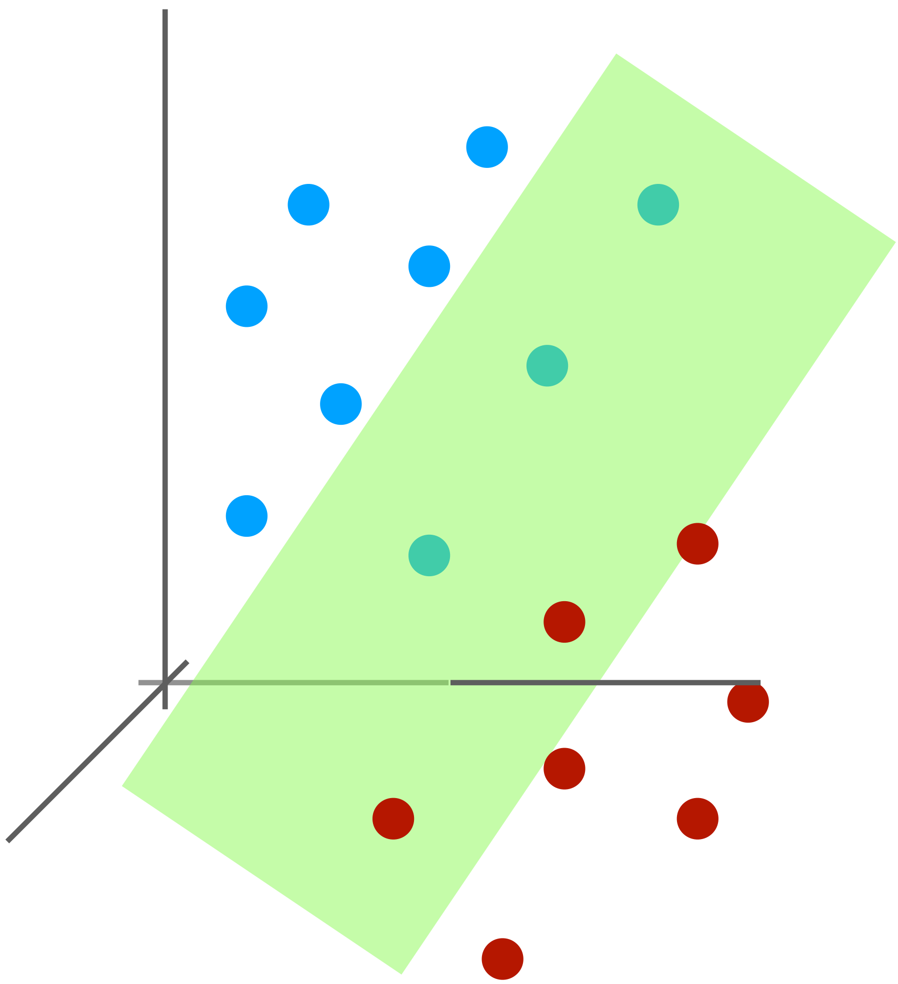

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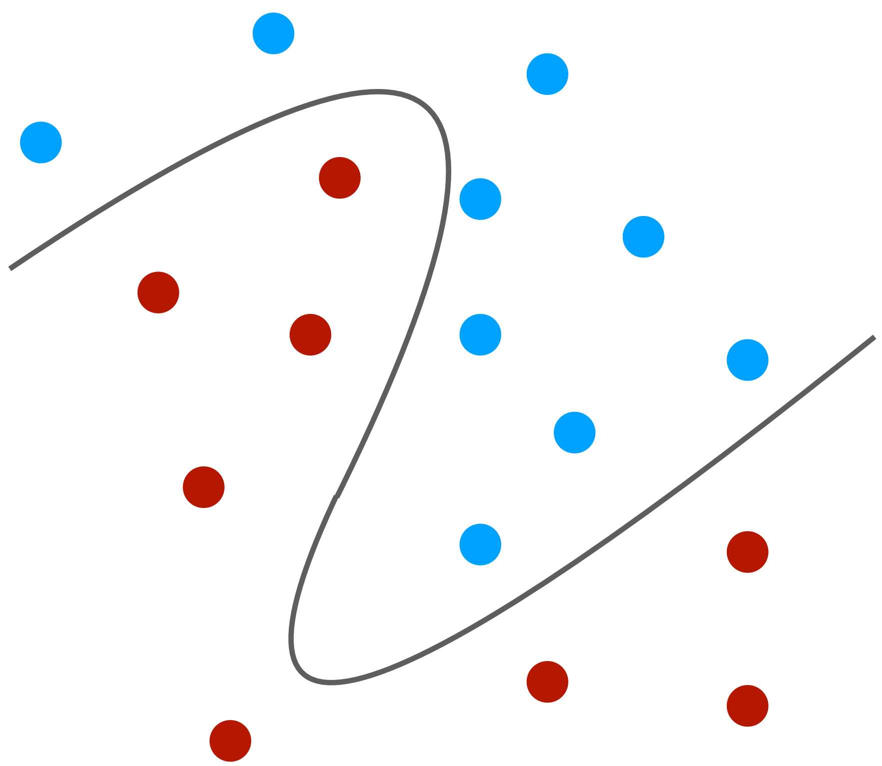
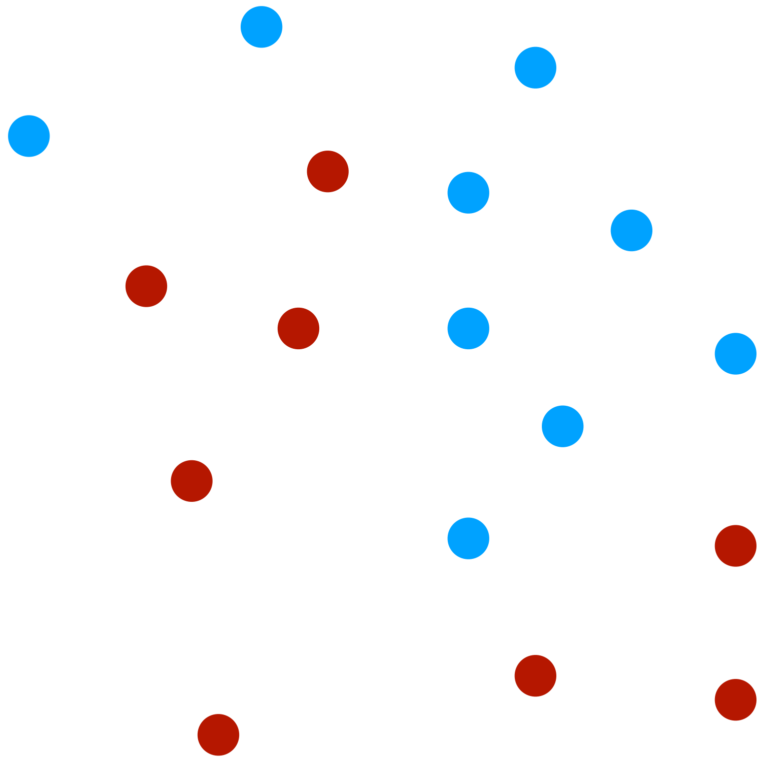


$$\sum_{|S|\leq 100} a_S \prod_{i\in S} x_i \geq 0$$

learn the coefficients a

Threshold degree

$$f : X \to \{0,1\}$$



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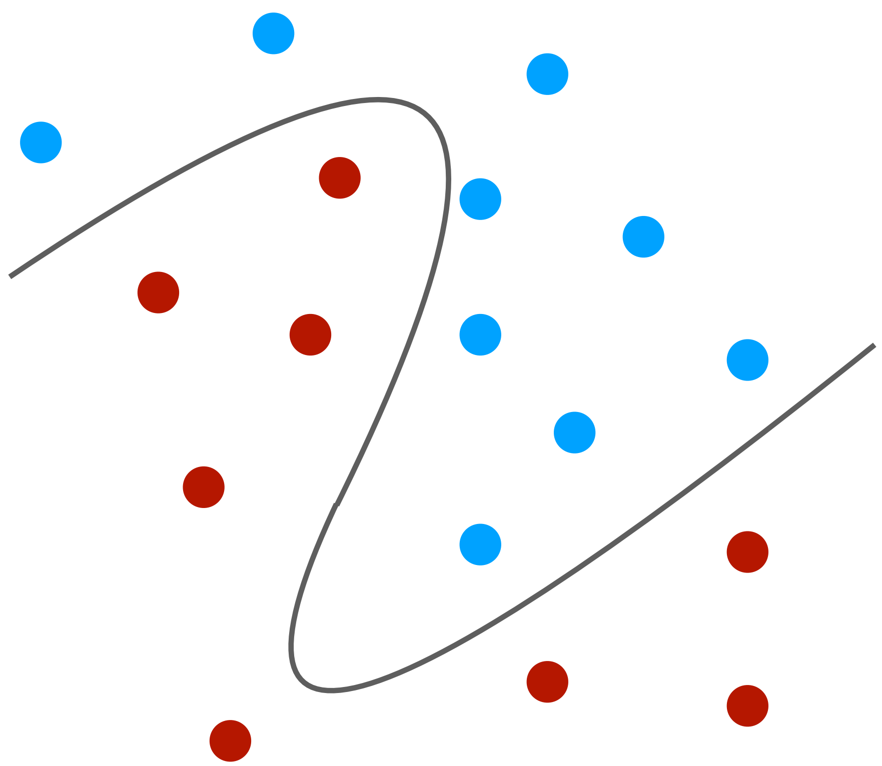

$$P(x) : X \to \mathbb{R},$$

$$P(x) = \sum_{|S| \le 100} a_S \prod_{i \in S} x_i .$$

Threshold degree

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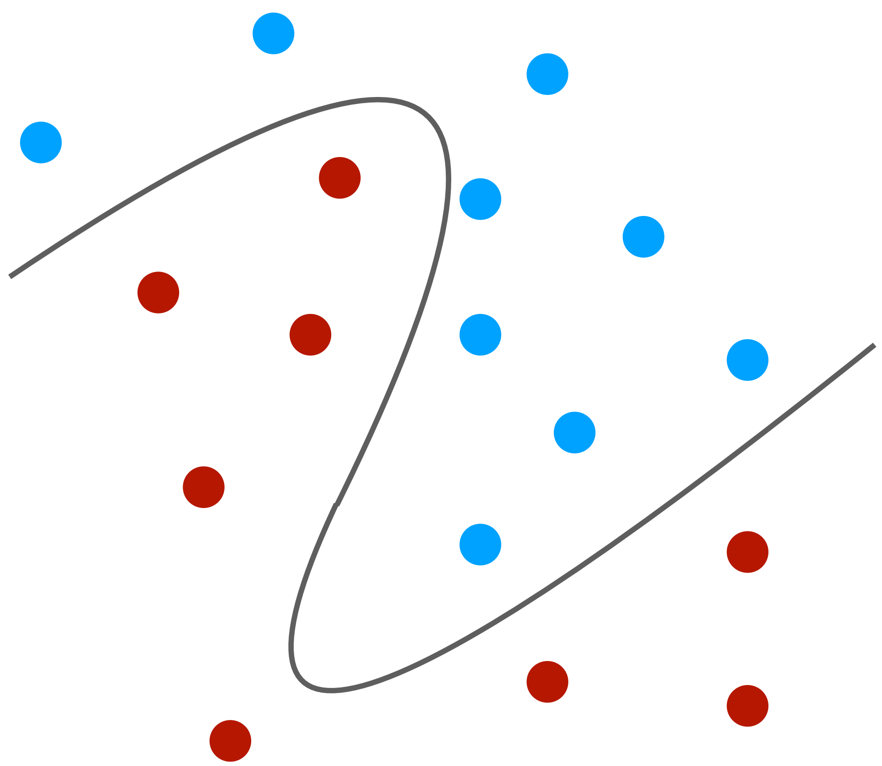

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P "sign represents" f

$$f(x) = 1 \iff P(x) < 0,$$

$$f(x) = 0 \iff P(x) > 0.$$

Threshold degree

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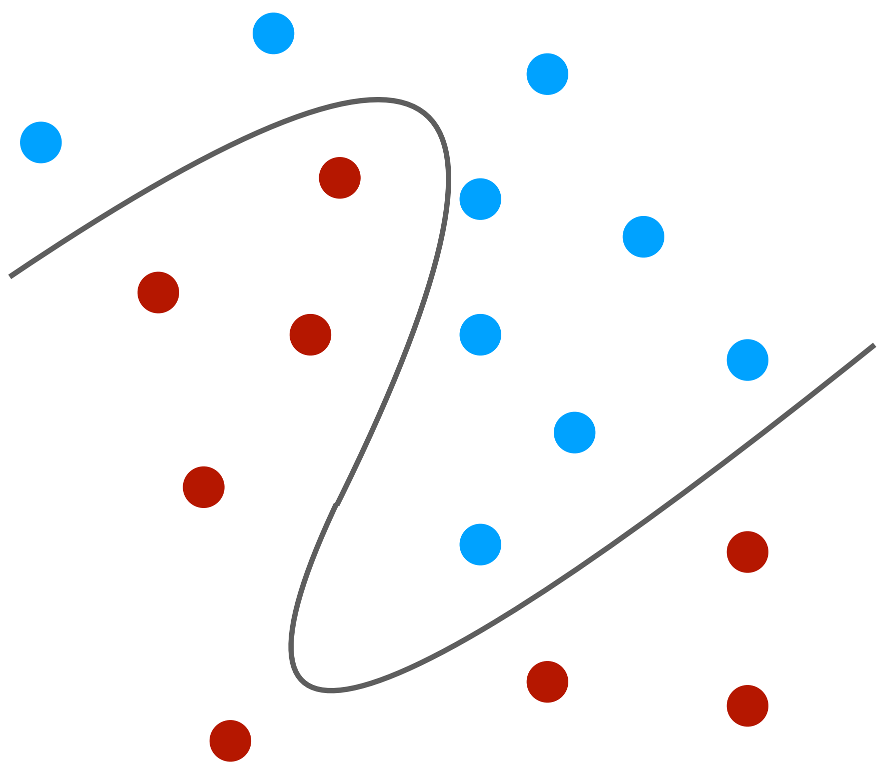

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$$f(x) = 1 \iff P(x) < 0,$$

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Or, simply

$$(-1)^{f(x)} P(x) > 0.$$

Threshold degree

$f : X \to \{0,1\}$

Definition.
$\deg_{\pm}(f) = \min\{\deg p :$
$p(x) \cdot (-1)^{f(x)} > 0$ for all $x \in X\}$.

Threshold degree

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Example: the AND function

$$\mathrm{AND}(11111) = 1, \mathrm{AND}(11011) = 0,$$

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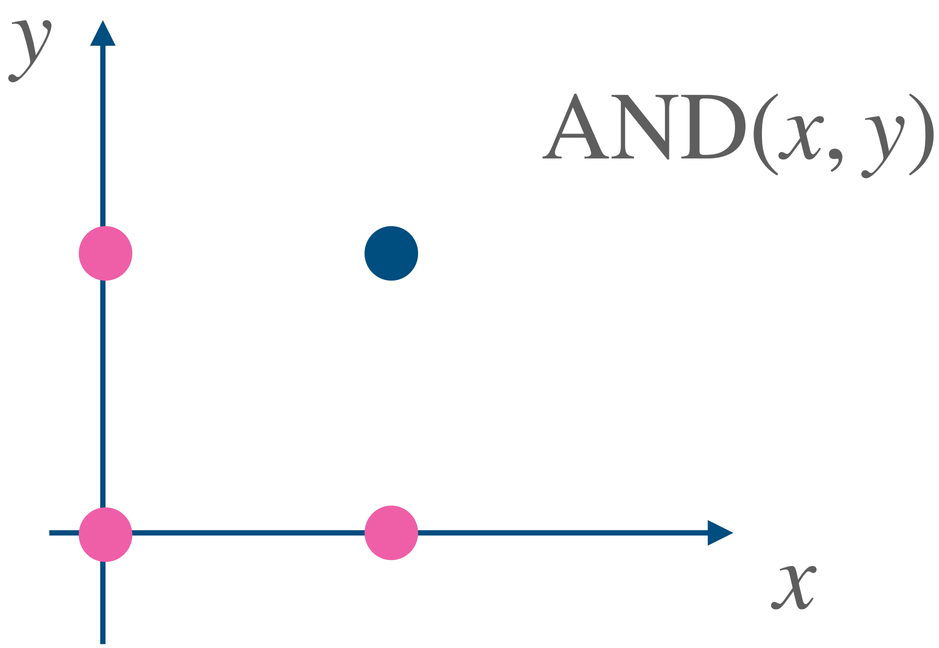

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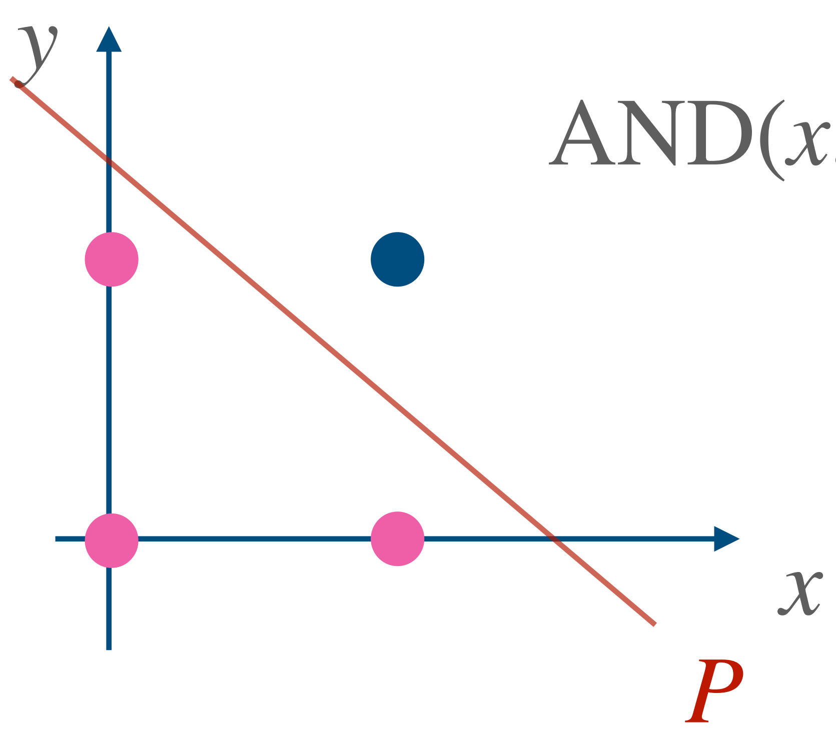

$$(n - 1/2 - x_1 - x_2 - \cdots - x_n)$$

Threshold degree

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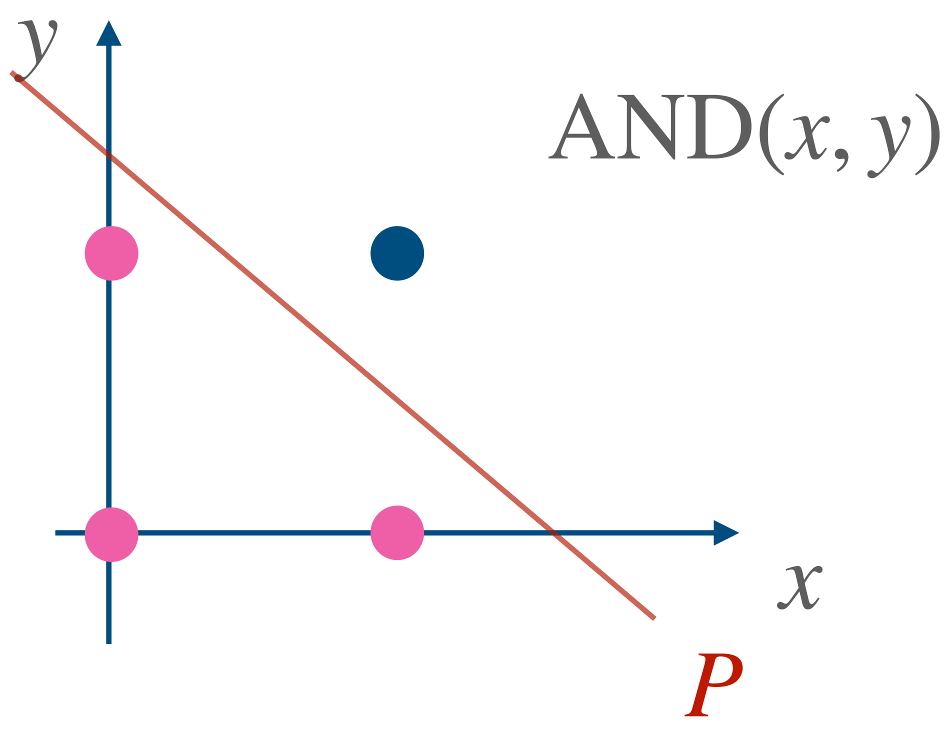

$$(-1)^{\mathrm{AND}(x)} \cdot (n - 1/2 - x_1 - x_2 - \cdots - x_n) > 0$$

Threshold degree

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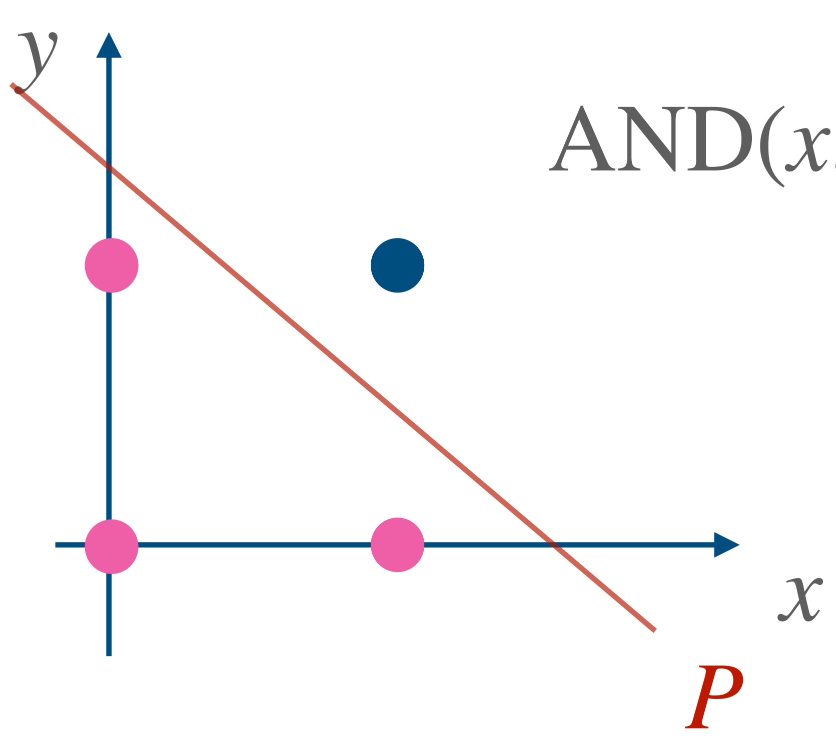

$\text{AND}(x, y)$

$$(-1)^{\text{AND}(x)} \cdot (n - 1/2 - x_1 - x_2 - \cdots - x_n) > 0$$

$$\deg_{\pm}(\text{AND}(x)) = 1.$$

Threshold degree

Example: the MAJORITY function

$\mathrm{MAJ}(\mathrm{x}) = 1$ if there are more 1s in x than 0s

e.g. $$\begin{aligned}\mathrm{MAJ}(11100) &= 1,\\ \mathrm{MAJ}(10100) &= 0\end{aligned}$$

Threshold degree

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e.g. $\begin{aligned}\mathrm{MAJ}(11100) &= 1,\\ \mathrm{MAJ}(10100) &= 0\end{aligned}$

$$\deg_{\pm}(\mathrm{MAJ}) = 1,$$

$$(-1)^{\mathrm{MAJ}(x)} \cdot \left(\frac{n}{2} - \sum_i x_i\right) > 0.$$

Threshold degree

Example: the XOR function

$\mathrm{XOR(x)} = 1$ if there are odd 1s in x

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Threshold degree

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Threshold degree

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Fact.

$\deg_{\pm}(\mathrm{XOR}_{\mathrm{n}}) = n\,.$

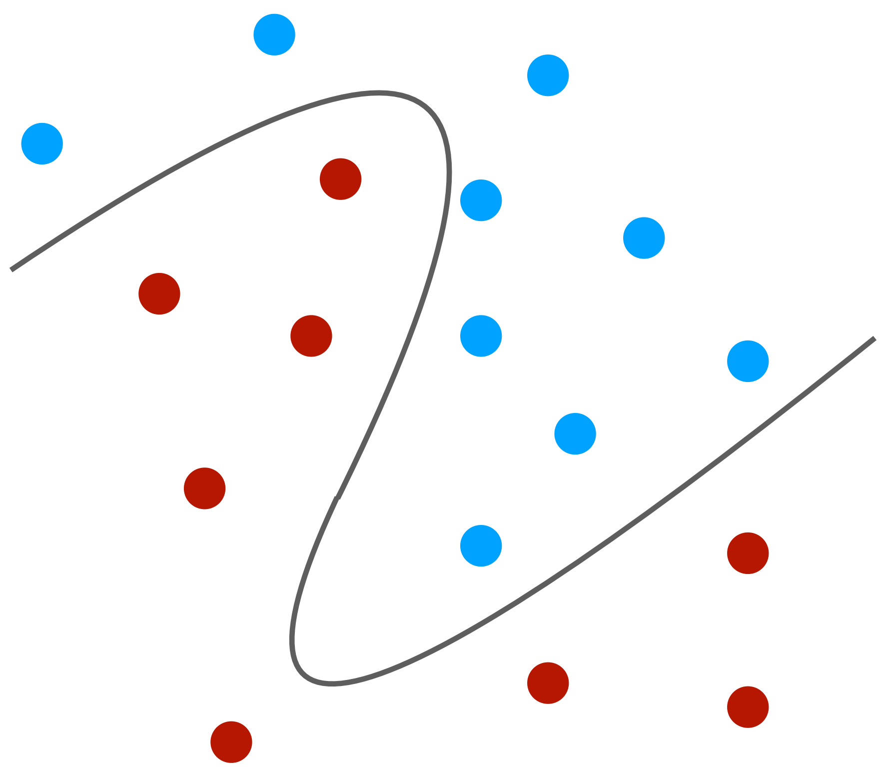
Threshold degree

$f : \{0,1\}^n \to \{0,1\}$

Fact.

$\deg_{\pm}(f) \leq n\,.$

Learning sparse polynomial



$$\sum_{|S| \geq n-100} a_S \prod_{i \in S} x_i \geq 0$$

learn the coefficients a

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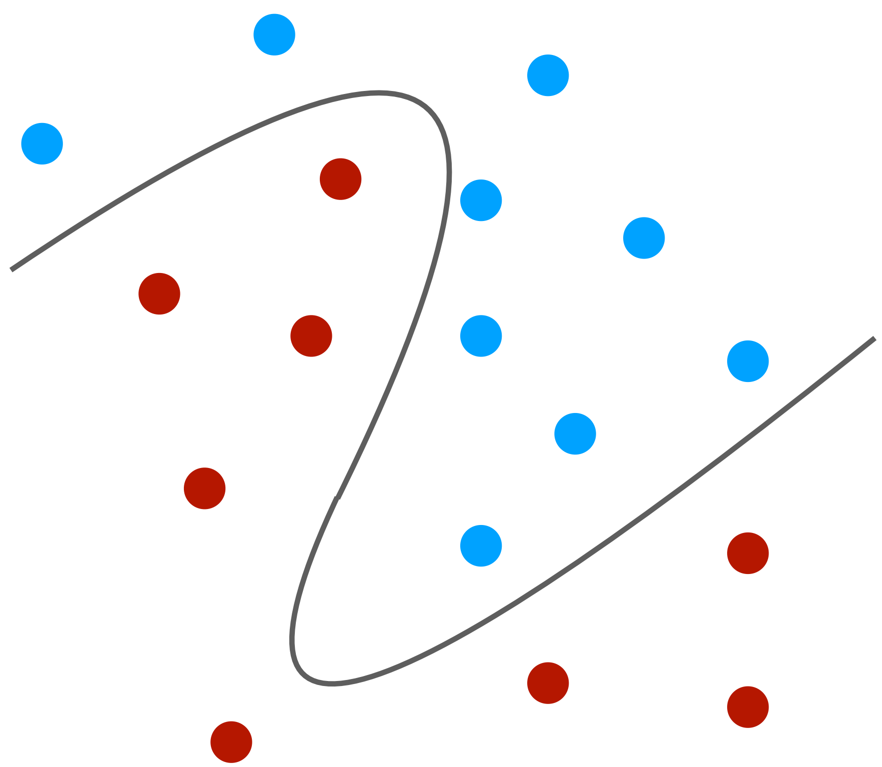

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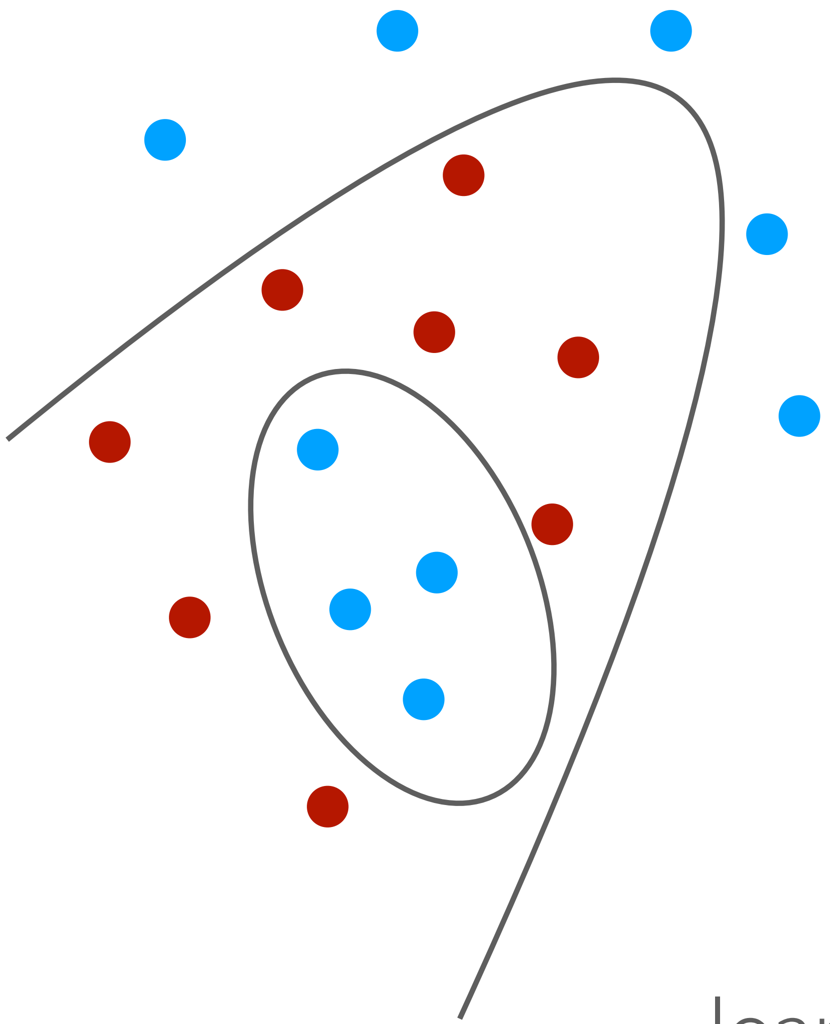
Learning more complicated function



$$\sum_i a_i \cdot f_i(x) \geq 0$$

learn the coefficients a

Sign rank

$f : X \times Y \to \{0,1\}$

Definition.

$$\mathrm{rk}_{\pm}(f) = \min\{\mathrm{rk} M : M(x,y) \cdot (-1)^{f(x,y)} > 0 \text{ for all } (x,y) \in X \times Y\}\,.$$

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Fact: $1 \leq \mathrm{rk}_{\pm}(f) \leq 2^n\,.$

Sign rank

Example:

$$M_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad M_4' = \begin{pmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 8 & 16 \\ 4 & 8 & 16 & 32 \\ 8 & 16 & 32 & 64 \end{pmatrix} - \begin{pmatrix} 9 & 9 & 9 & 9 \\ 9 & 9 & 9 & 9 \\ 9 & 9 & 9 & 9 \\ 9 & 9 & 9 & 9 \end{pmatrix}$$

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$$\mathrm{rk}(M_4') = 2,$$

Sign rank

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$$\mathrm{rk}(M_4) = 4, \qquad \mathrm{rk}(M_4') = 2,$$

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$$M_4' = \begin{pmatrix} - & - & - & - \\ - & - & - & + \\ - & - & + & + \\ - & + & + & + \end{pmatrix}$$

Sign rank

More generally,

$$M_k = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad \mathrm{rk}(M_k) = k\,.$$

$$M'_k = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ 2^{k-1} \end{pmatrix} \times \begin{pmatrix} 1 & 2 & \dots & 2^{k-1} \end{pmatrix} - (2^{k-1} + 1)J, \qquad \mathrm{rk}(M'_k) \leq 2.$$

Classic problem 1

Problem.
Is there $f \in \mathrm{AC}^0$ such that $\deg_{\pm}(f) = \Omega(n)$?

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reference	threshold degree	depth
Minsky-Papert 69	$\Omega(n^{1/3})$	2
O'Donnell-Servedio 03	$\Omega(n^{1/3} \log^{\frac{2(k-2)}{3}} n)$	k
Sherstov 14	$\Omega(n^{\frac{k-1}{2k-1}})$	k
Sherstov 15	$\Omega(\sqrt{n})$	4
Bun-Thaler 18	$\tilde{\Omega}(\sqrt{n})$	3

Classic problem I

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Optimal

Classic problem II

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1. AC^0 has the maximal complexity!

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   [Paturi, Simon 86]
   $\mathrm{UPP}(F) = \log_2(\mathrm{rk}_\pm(F)) \pm 1.$

Our result

1. AC^0 has the maximal complexity!

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[Paturi, Simon 86]

$$\mathrm{UPP}(F) = \log_2(\mathrm{rk}_\pm(F)) \pm 1.$$

Corollary.

$$\mathrm{UPP}(\mathrm{AC}^0) = \Omega(n^{0.99}) .$$

Part I.
Threshold Degree

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a. Hardness amplification

b. Compressing inputs

c. Transferring mass

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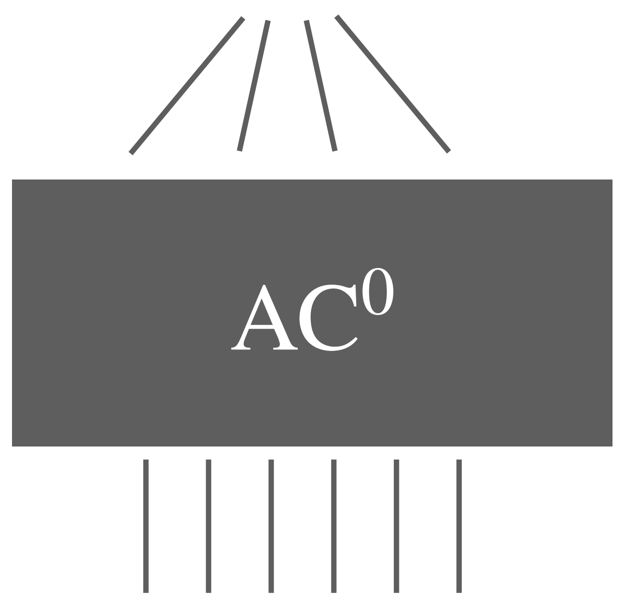
Then $F = f$ $\quad \deg_{\pm}(F) = N^{1-\frac{\epsilon}{\epsilon+1}}$



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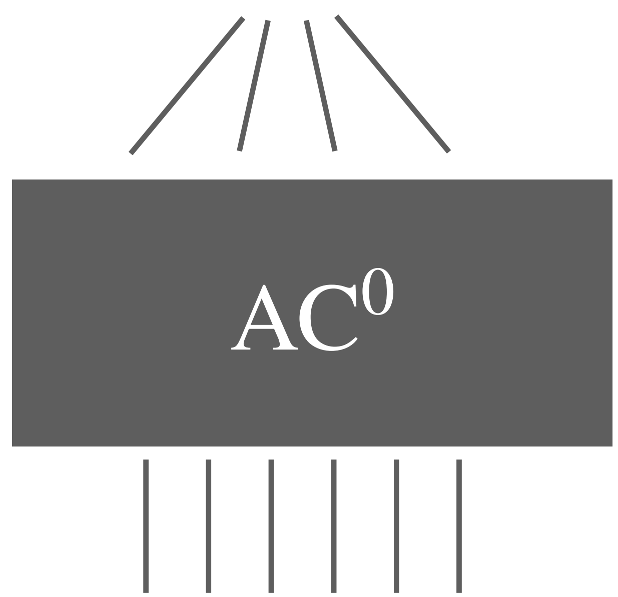

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Thereom (Sherstov)

$\deg_{\pm}(f \circ g) \geq \deg_{\pm}(f)\deg_{\pm}(g)$.

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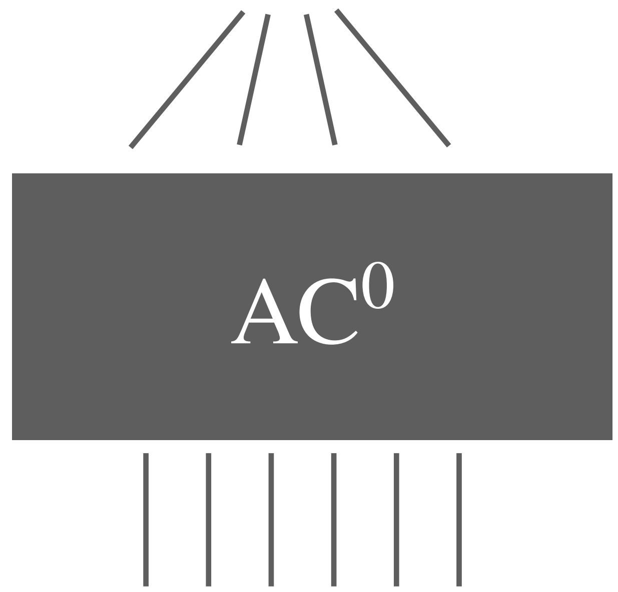

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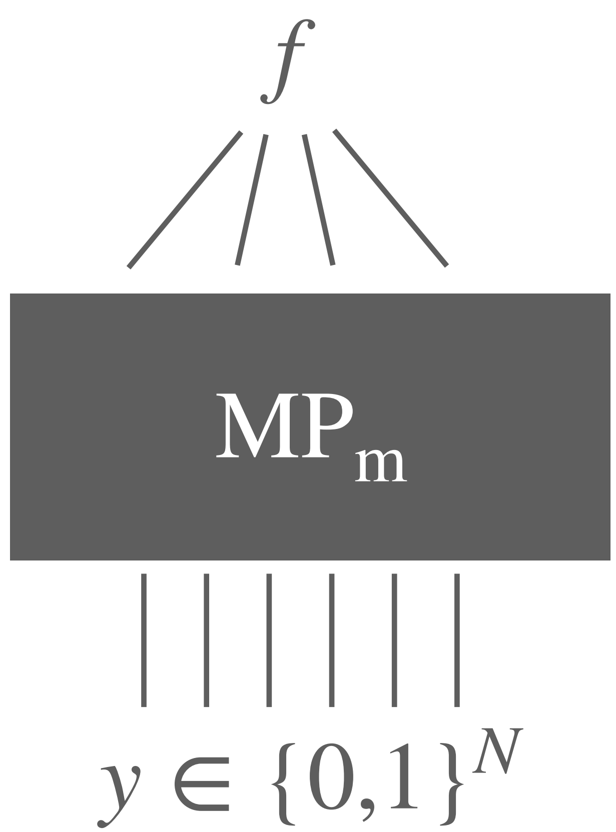
Thereom (Sherstov)

$\deg_{\pm}(f \circ g) \geq \deg_{\pm}(f)\deg_{\pm}(g)$.

$f \circ g(x) := f(g(x_1), g(x_2), \ldots, g(x_n))$

What to compose?

Given $f : \{0,1\}^n \rightarrow \{0,1\}, \qquad \deg_{\pm}(f) = n^{1-\epsilon}$



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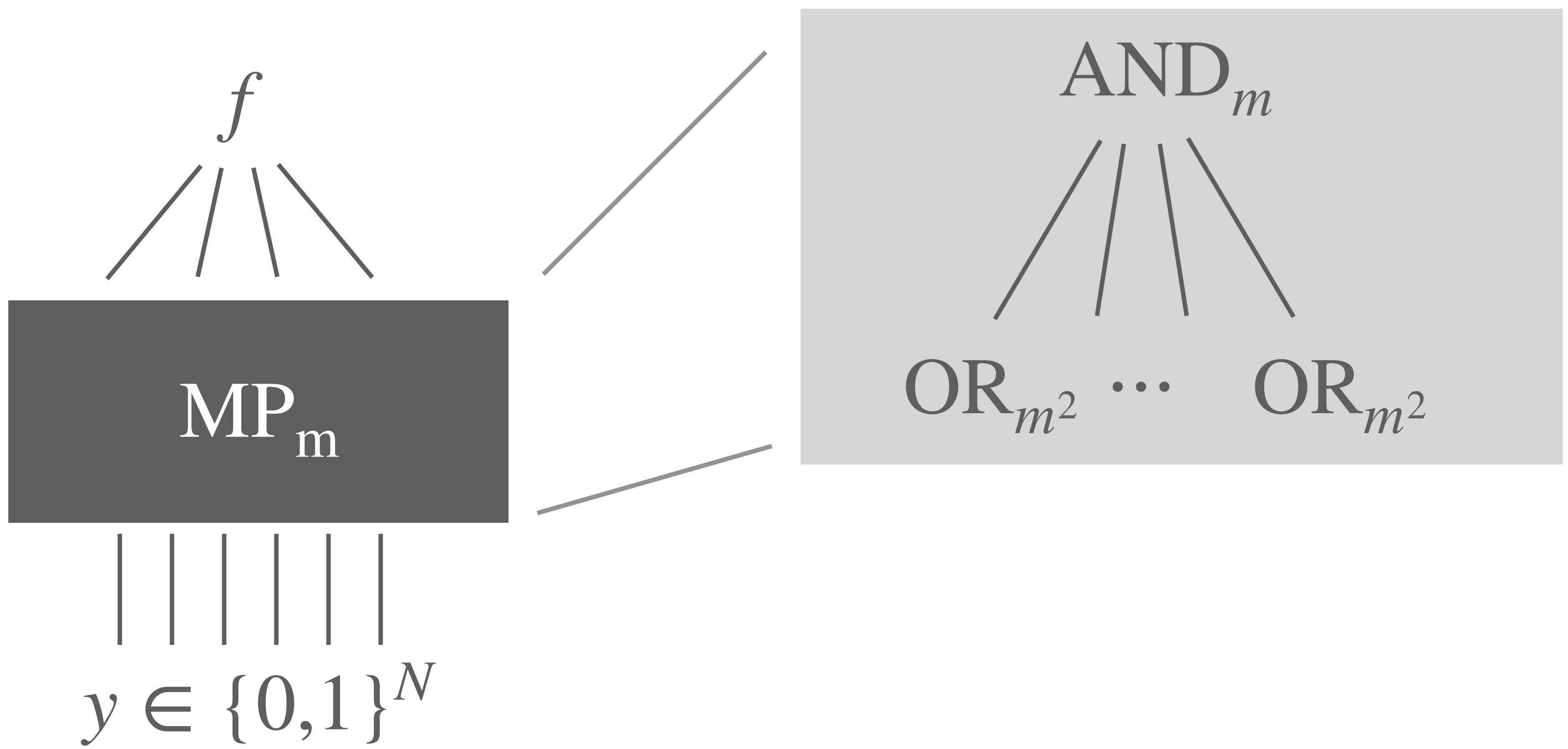

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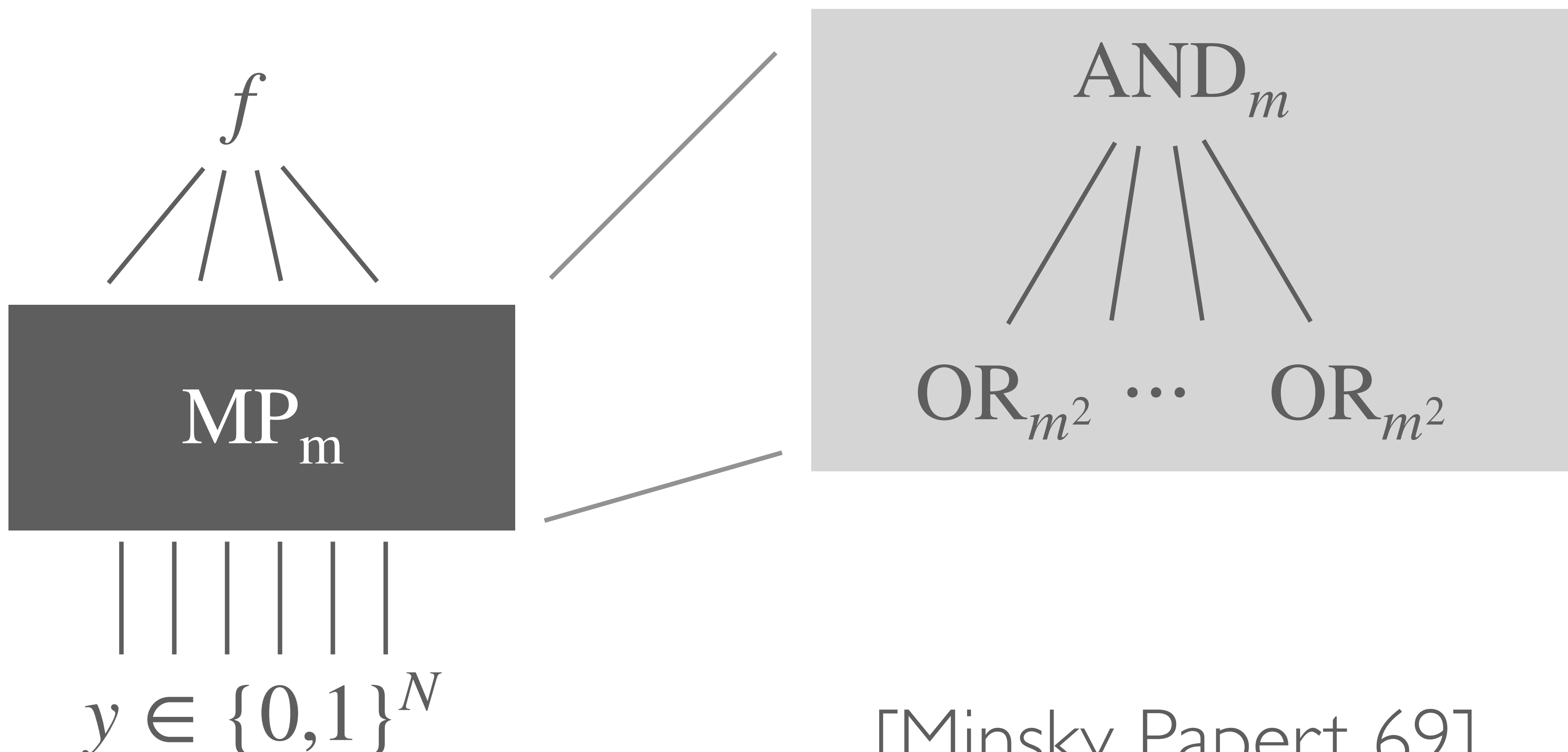
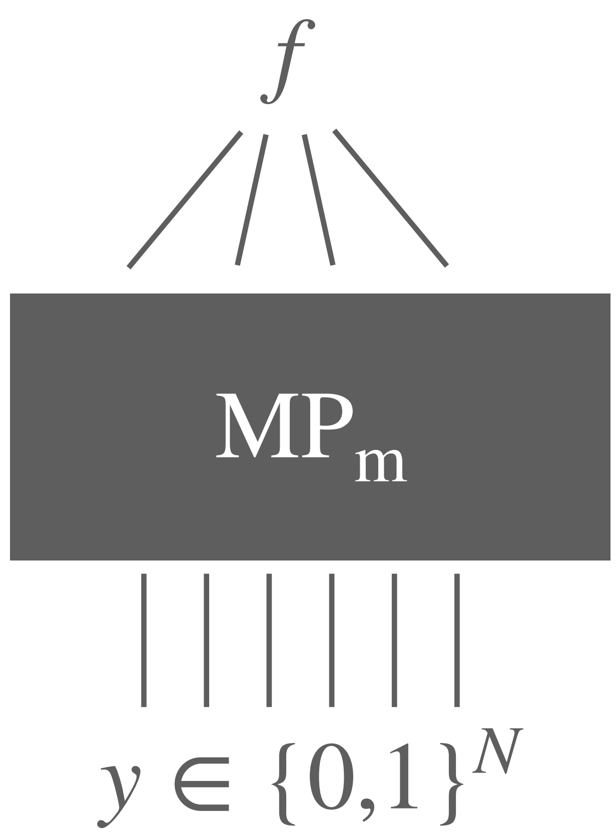


[Minsky, Papert 69]
$\deg_{\pm}(\mathrm{MP}_m) \geq m$.

Threshold degree of compose function

Given $f : \{0,1\}^n \to \{0,1\}$, $\qquad \deg_{\pm}(f) = n^{1-\epsilon}$



$$\deg_{\pm}(f \circ \mathrm{MP}_m) \geq n^{1-\epsilon} \cdot m$$

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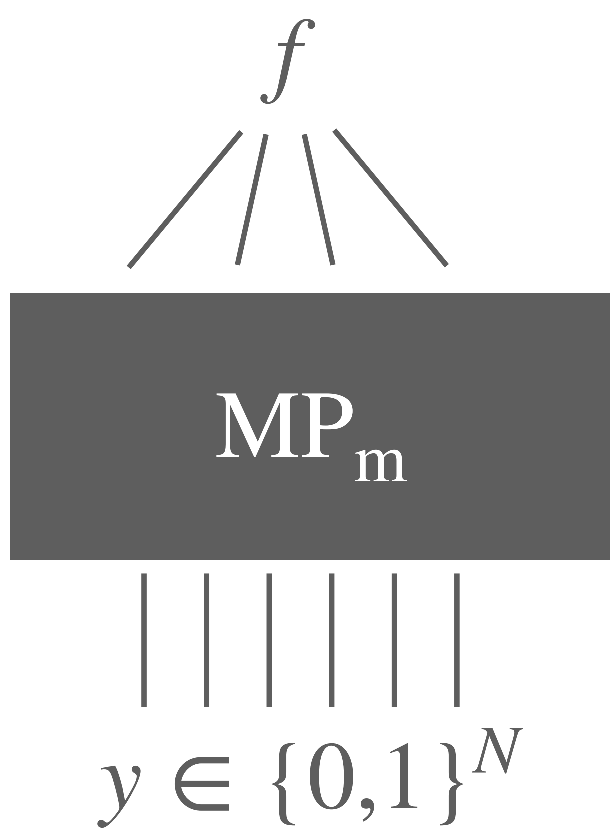


$$\deg_{\pm}(f \circ \mathrm{MP}_m) \geq n^{1-\epsilon} \cdot m$$

But…

$$N = n \cdot m^3$$

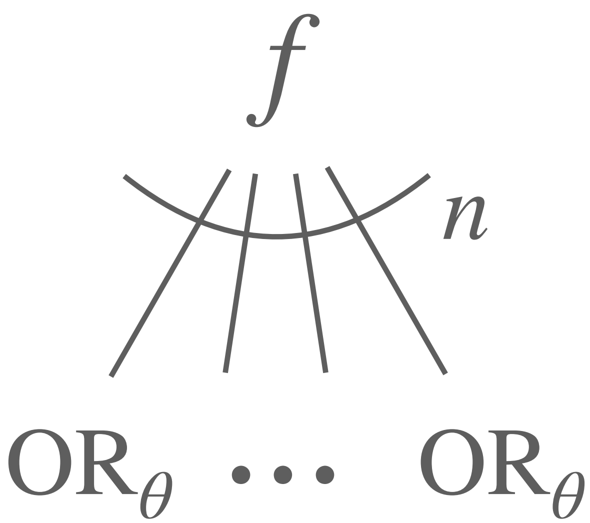
Part 1.
Threshold Degree

a. Hardness amplification

b. Compressing inputs

c. Transferring mass

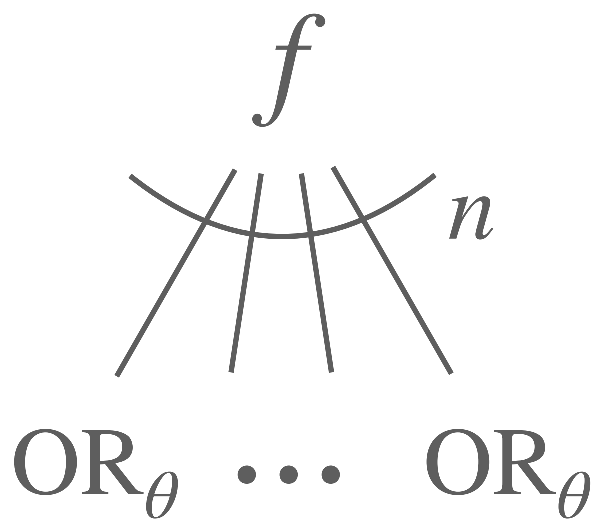
Compression: input transformation



	0	1	1	0	0	1	1
	0	0	1	1	0	1	0
	1	0	1	0	1	0	1
θ	0	1	0	1	1	0	0
	0	0	1	0	0	1	1
	1	1	0	0	0	1	0
	1	1	0	1	1	0	0
	0	0	1	1	0	0	1

n

Compression: input transformation



θ

0	1	1	0	0	1	1
0	0	1	1	0	1	0
1	0	1	0	1	0	1
0	1	0	1	1	0	0
0	0	1	0	0	1	1
1	1	0	0	0	1	0
1	1	0	1	1	0	0
0	0	1	1	0	0	1

n

Restrict

$$\xrightarrow[\{0,1\}^{\theta\times n}|_{\leq\theta}]{}$$

0	0	0	0	0	0	0
0	0	0	1	0	0	0
0	0	1	0	0	0	0
0	0	0	0	1	0	0
0	0	0	0	0	1	0
0	1	0	0	0	0	0
1	0	0	0	0	0	0
0	0	0	0	1	0	0

Block composition followed by compression

$$\mathrm{AND}_m$$

$$\mathrm{OR}_{m^2} \quad \cdots \quad \mathrm{OR}_{m^2}$$

[Minsky-Papert 69]

$$\deg_{\pm}(\mathrm{MP}_m) = \Omega(m)\,.$$

[Bun-Thaler 18]

$$\deg_{\pm}(\mathrm{MP}_m|_{\leq m^2}) = \tilde{\Omega}(m)\,.$$

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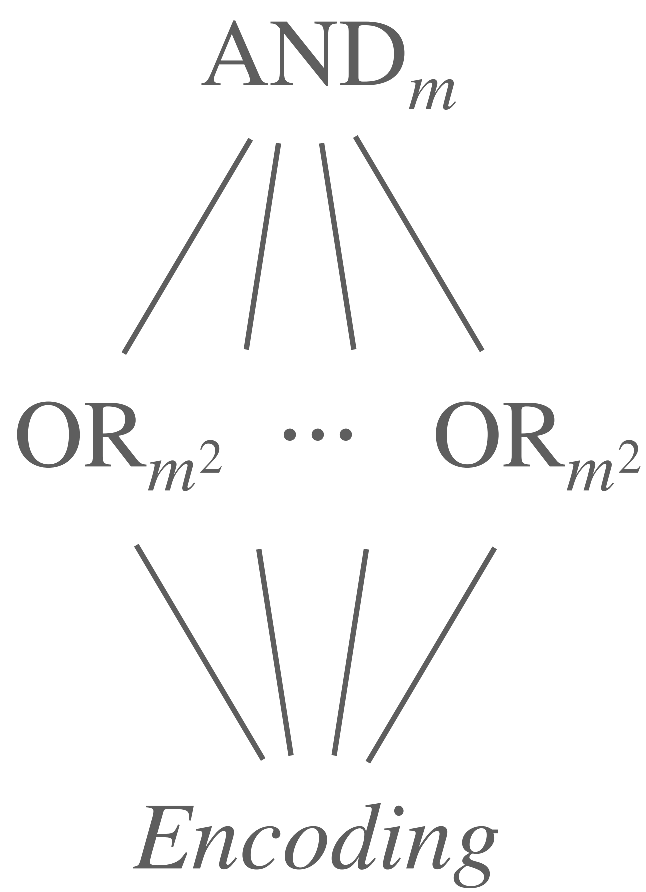

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$$\deg_{\pm}(F) = \tilde{\Omega}(m)\,.$$

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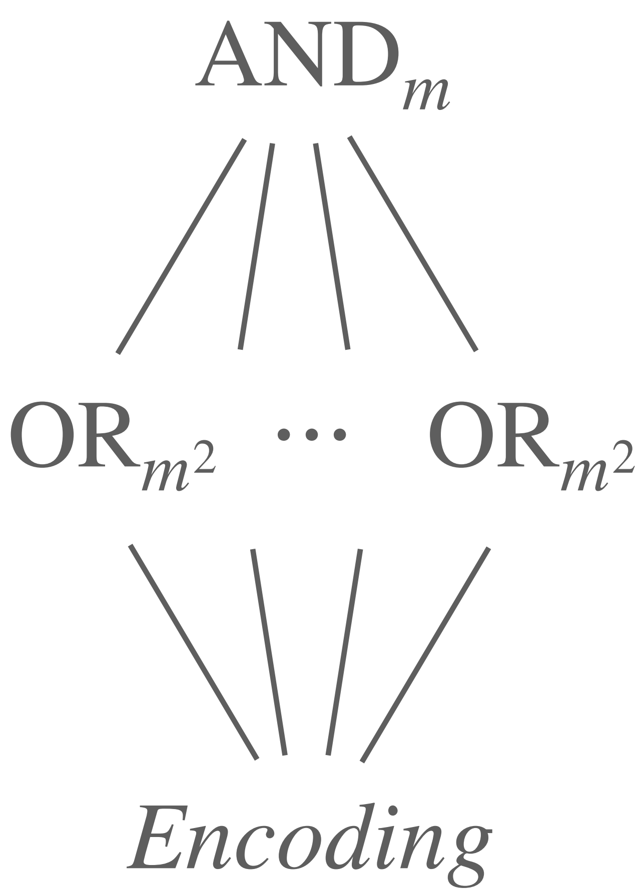

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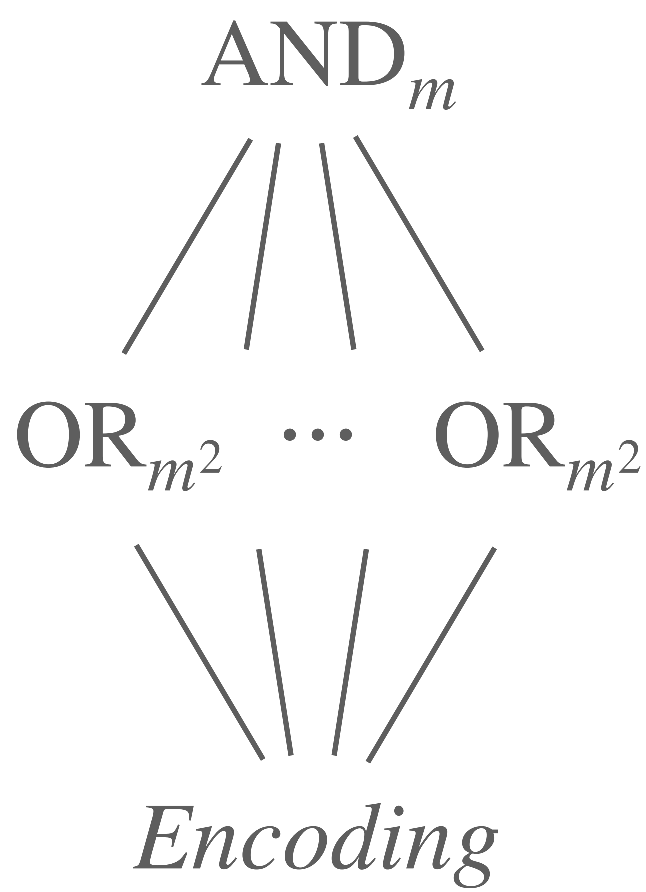

[Minsky-Papert 69] $\Omega(n^{1/3})$

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Threshold degree of compose function

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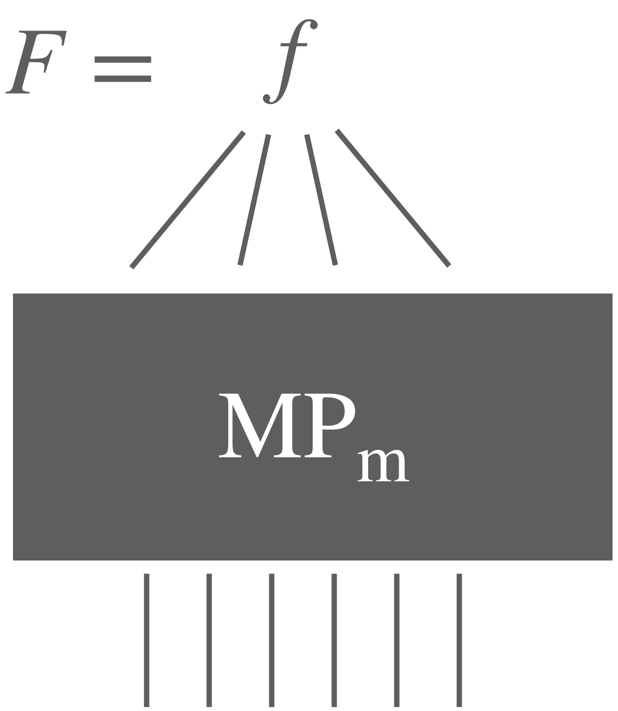
$F =$



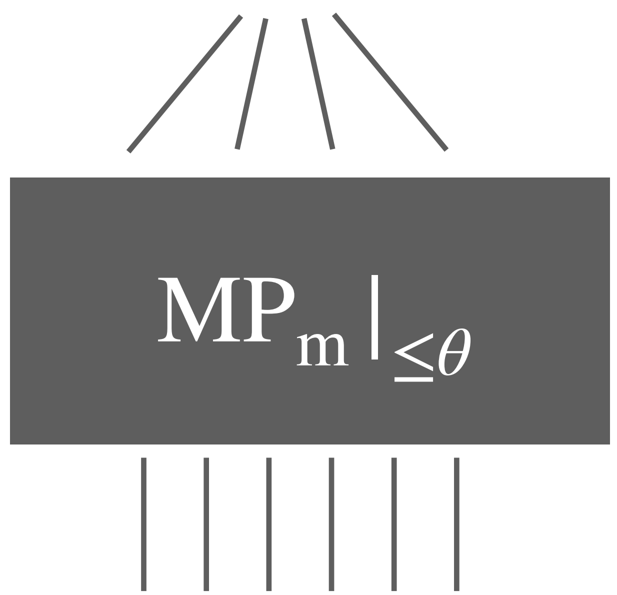
$y \in \{0,1\}^N$

$$\deg_{\pm}(f \circ \mathrm{MP}_m) \geq n^{1-\epsilon} \cdot m$$

Threshold degree of compose function

Given $f: \{0,1\}^n \rightarrow \{0,1\}, \qquad \deg_{\pm}(f) = n^{1-\epsilon}$

$F = \quad f$



$y \in \{0,1\}^N$

$(f \circ \mathrm{MP_m})|_{\leq\theta}$

$\deg_{\pm}(\cancel{f \circ \mathrm{MP}_m}) \geq n^{1-\epsilon} \cdot m$

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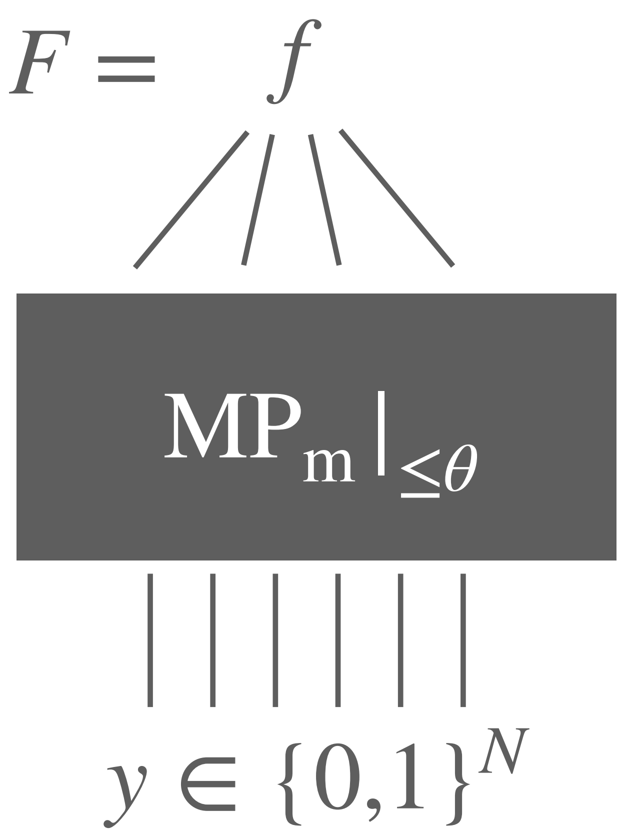


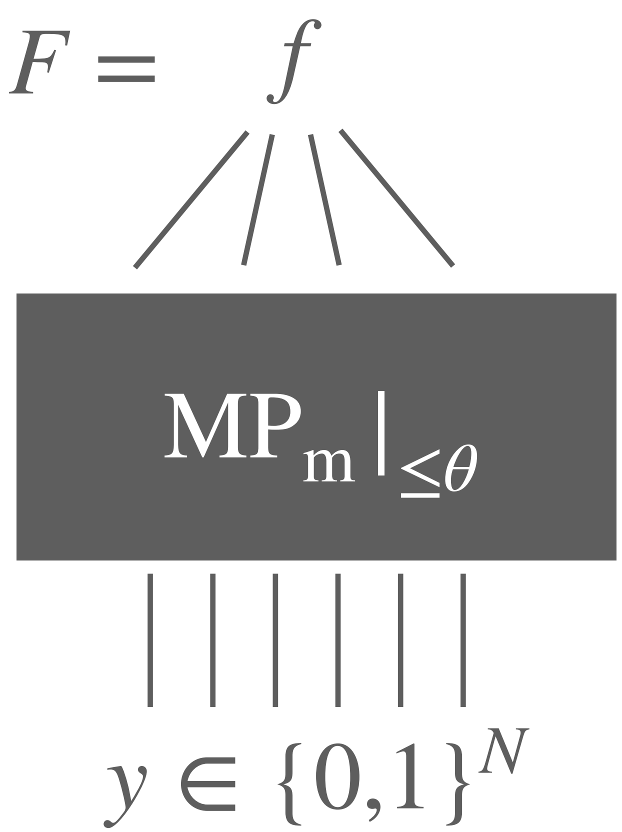
$$\deg_{\pm}\Big(\overset{(f\circ \mathrm{MP_m})|_{\le\theta}}{\cancel{f\circ \mathrm{MP}_m}}\Big) \ge n^{1-\epsilon}\cdot m$$

Now,

$$N = \tilde{O}(\theta)$$

Threshold degree of compose function

Given $f : \{0,1\}^n \to \{0,1\},\qquad \deg_\pm(f) = n^{1-\epsilon}$



$f \circ \mathrm{MP_m}|_{\leq\theta}$

$\deg_\pm(\cancel{f \circ \mathrm{MP}_m}) \geq n^{1-\epsilon} \cdot m$

$N = \tilde{O}(\theta)$

"Should this hold?"

Part 1.
Threshold Degree

a. Hardness amplification

b. Compressing inputs

c. Transferring mass

Dual characterization

$f : X \to \{0,1\}$

$\deg_{\pm}(f) \geq d \quad \Longleftarrow$

Exits some “witness" ψ with domain X

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L.P.

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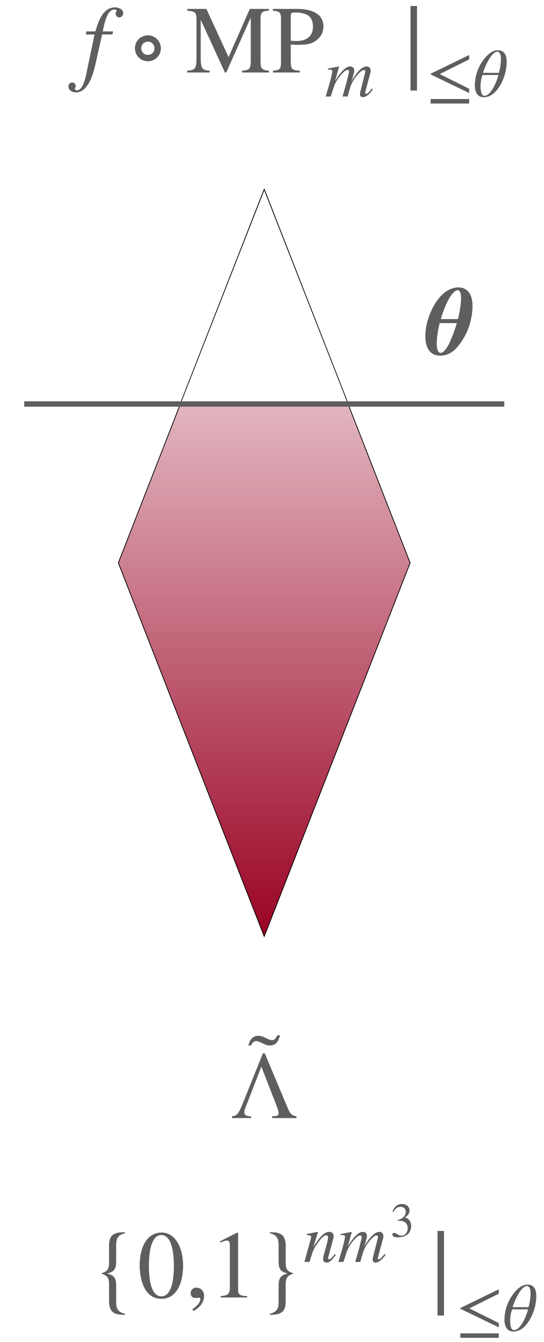
Transferring mass

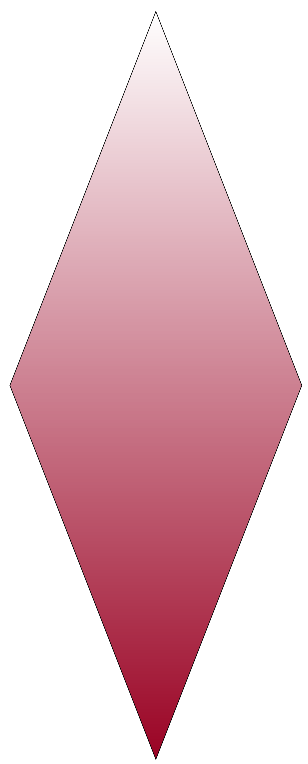
$$f \circ \mathrm{MP}_m |_{\leq \theta}$$

Transferring mass



Transferring mass

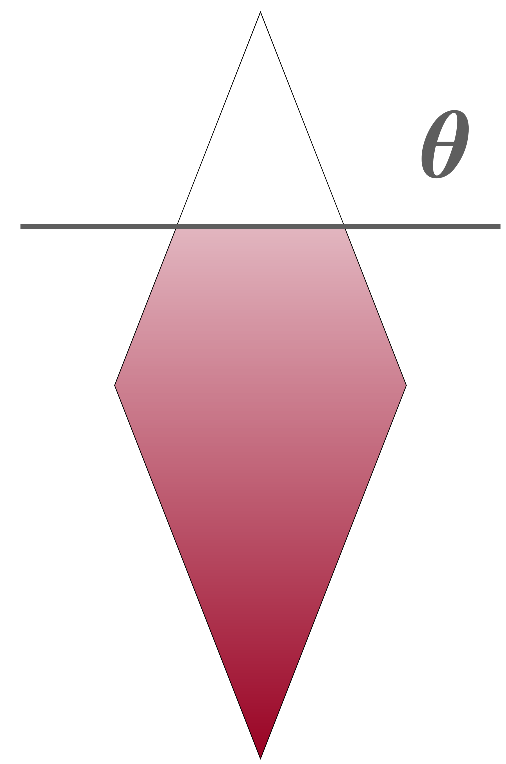
$f \circ \mathrm{MP}$



Λ

$\{0,1\}^{nm^3}$

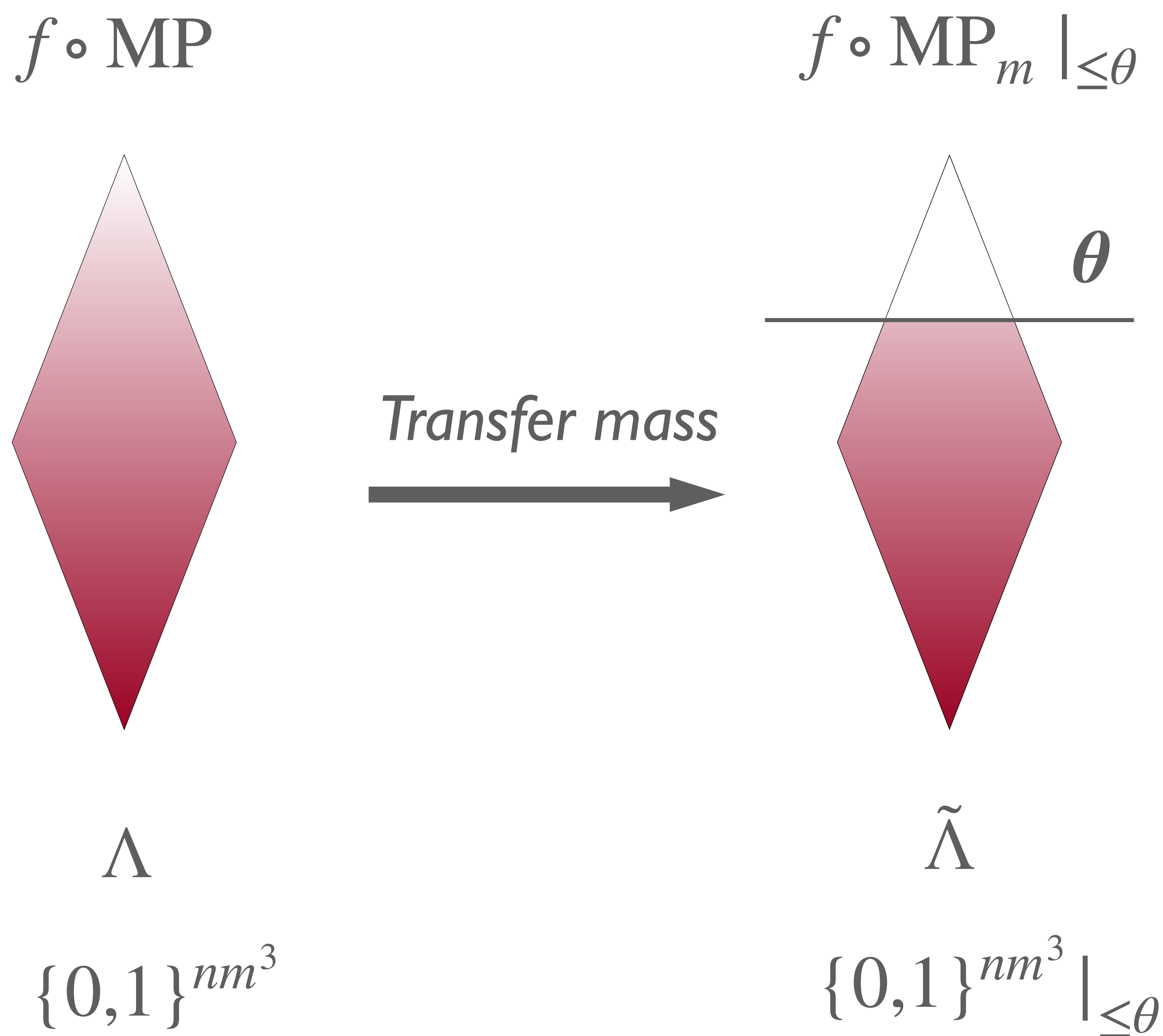
$f \circ \mathrm{MP}_m|_{\leq\theta}$



$\tilde{\Lambda}$

$\{0,1\}^{nm^3}|_{\leq\theta}$

Transferring mass



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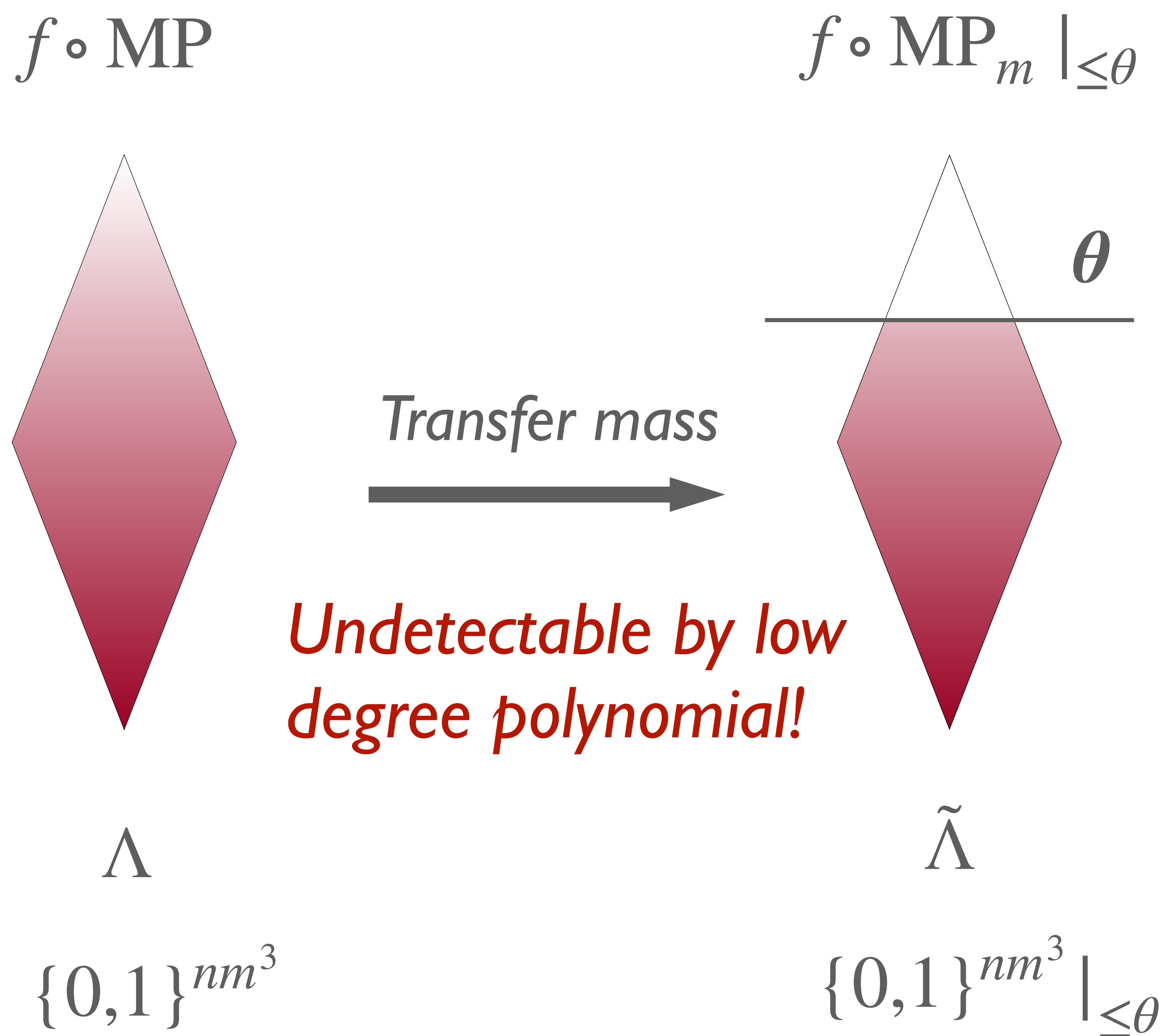
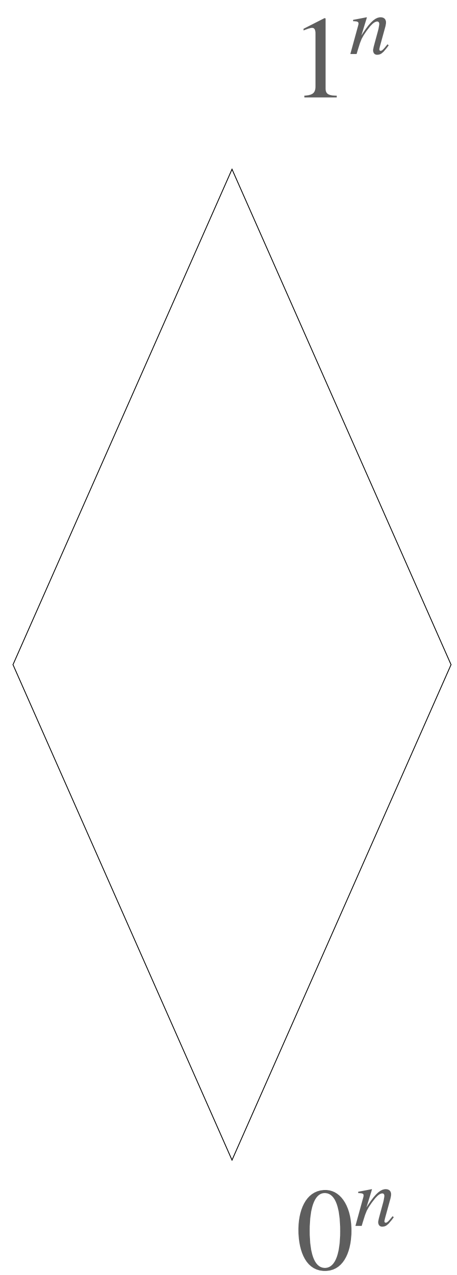


How do we transfer mass: Corrector function



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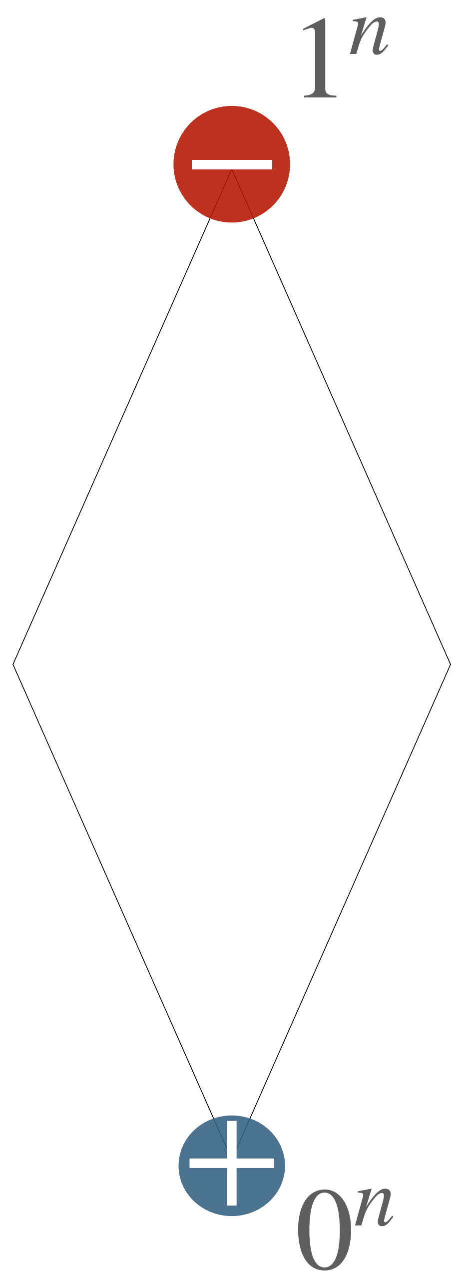

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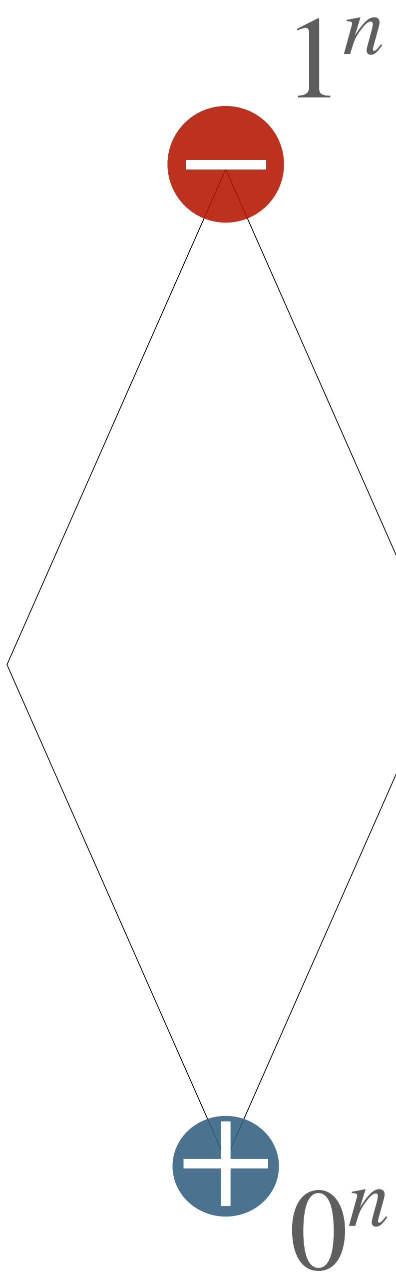
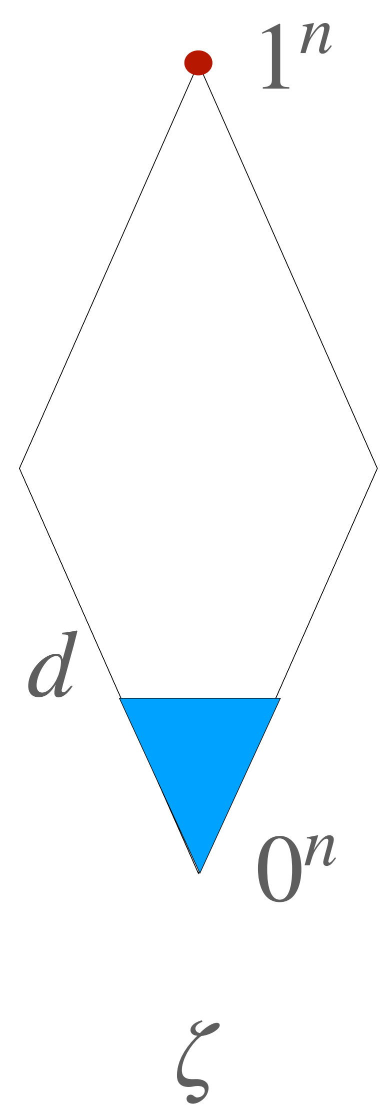


[Razborov, Sherstov 07]



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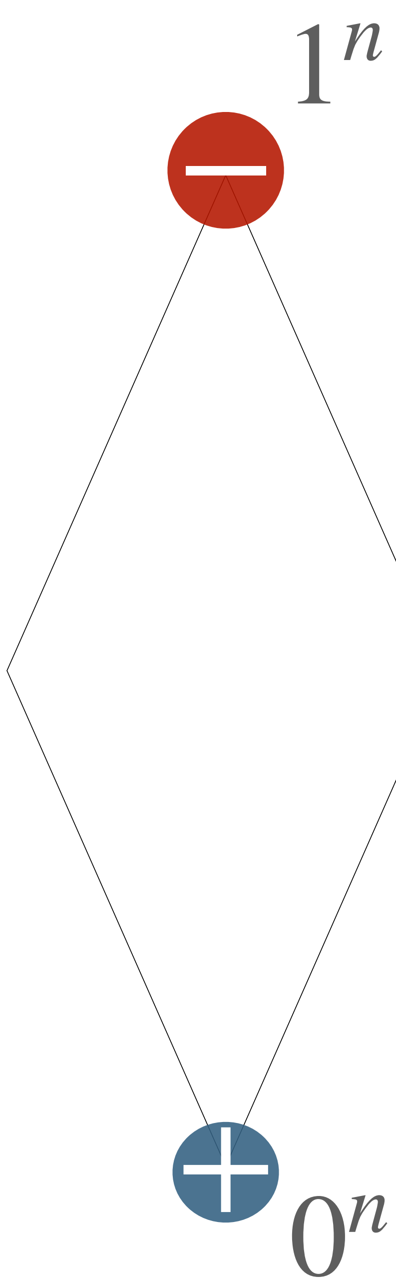

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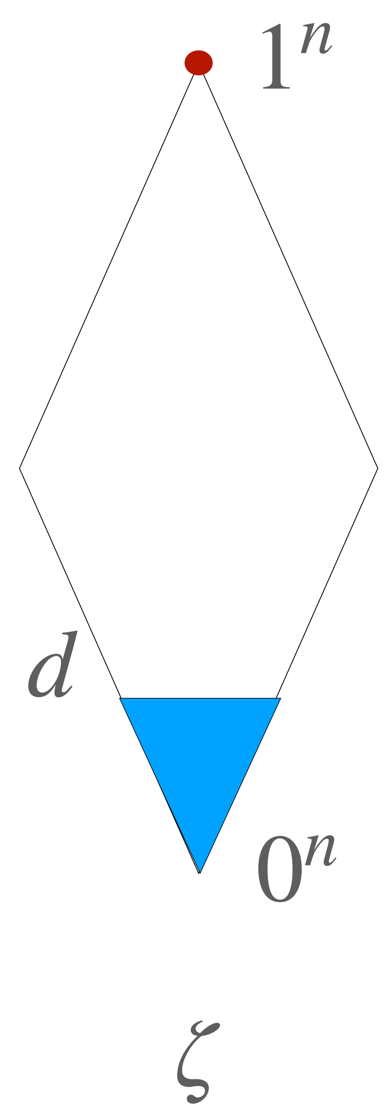

1. vanishes in the middle.

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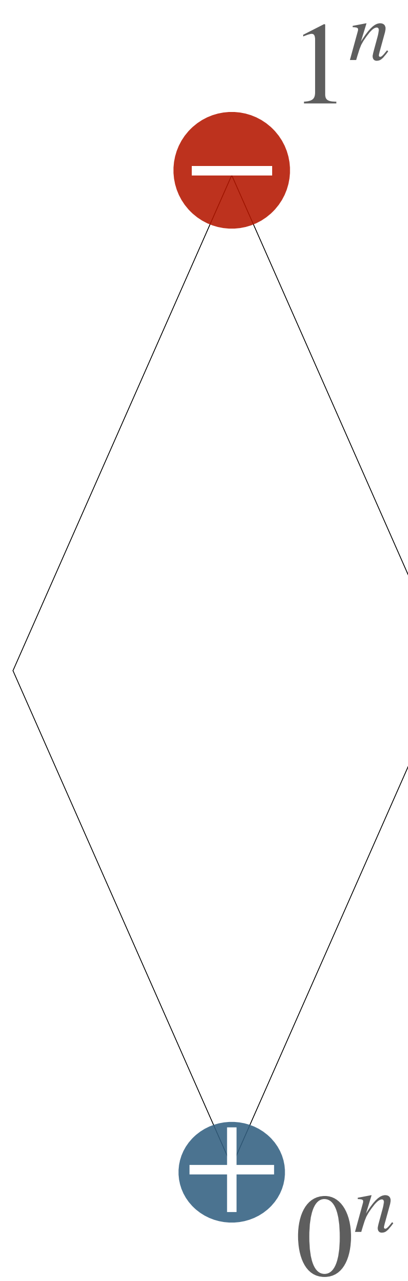

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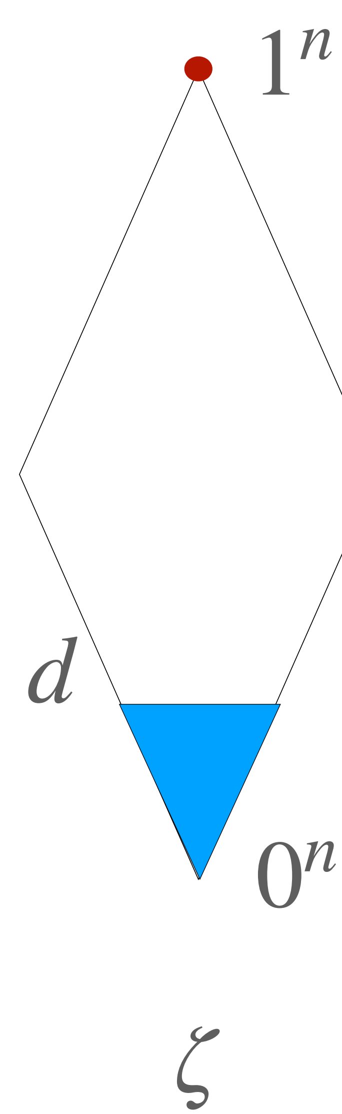
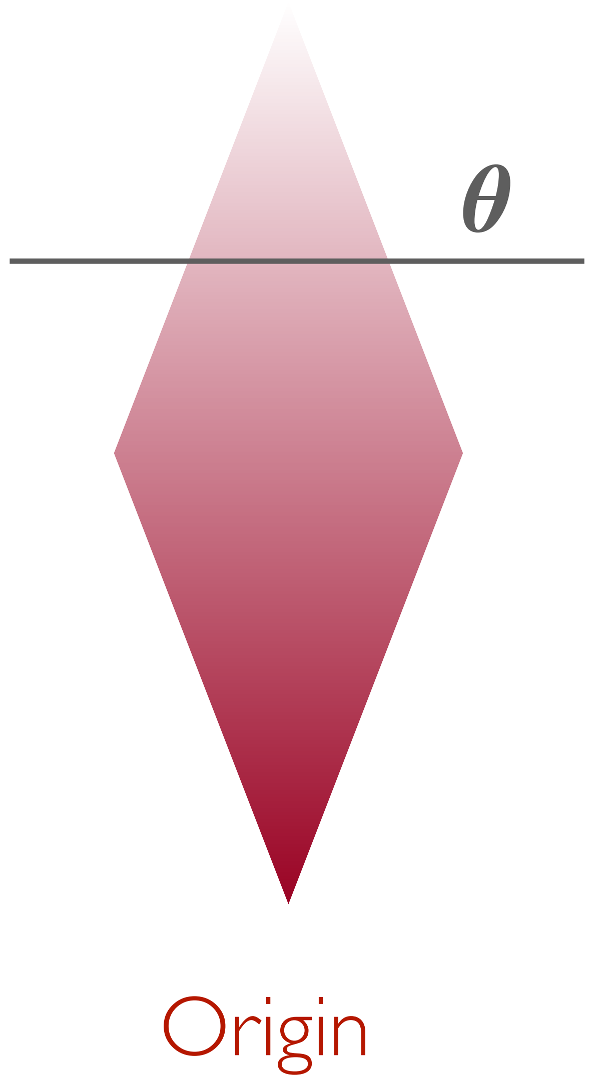


1. vanishes in the middle.

2. has no components of degree $\leq d$.

Previous work [BT 17]: Transfer mass to the origin



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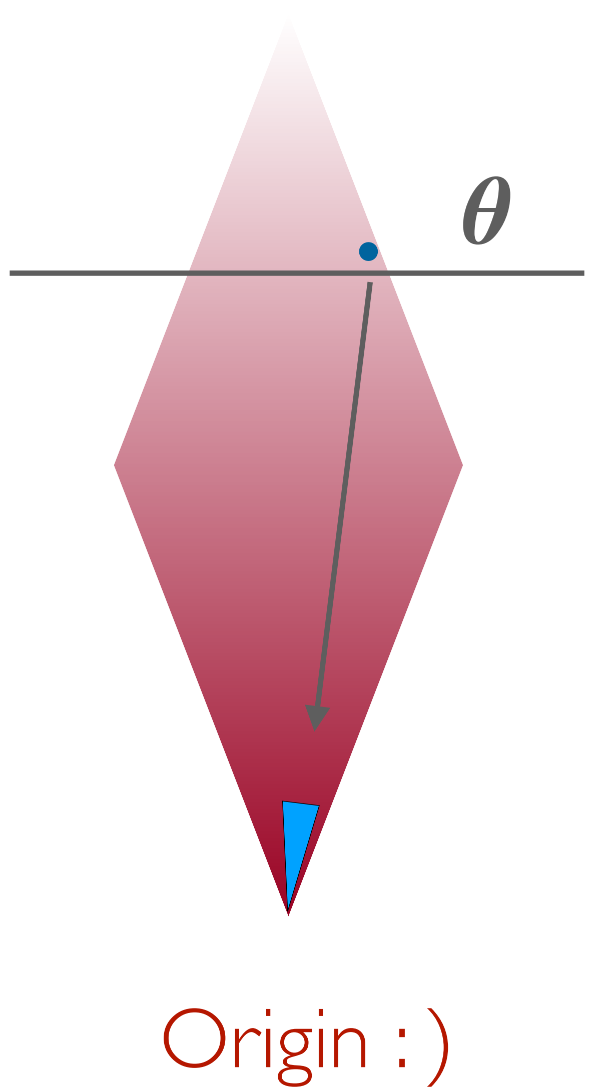

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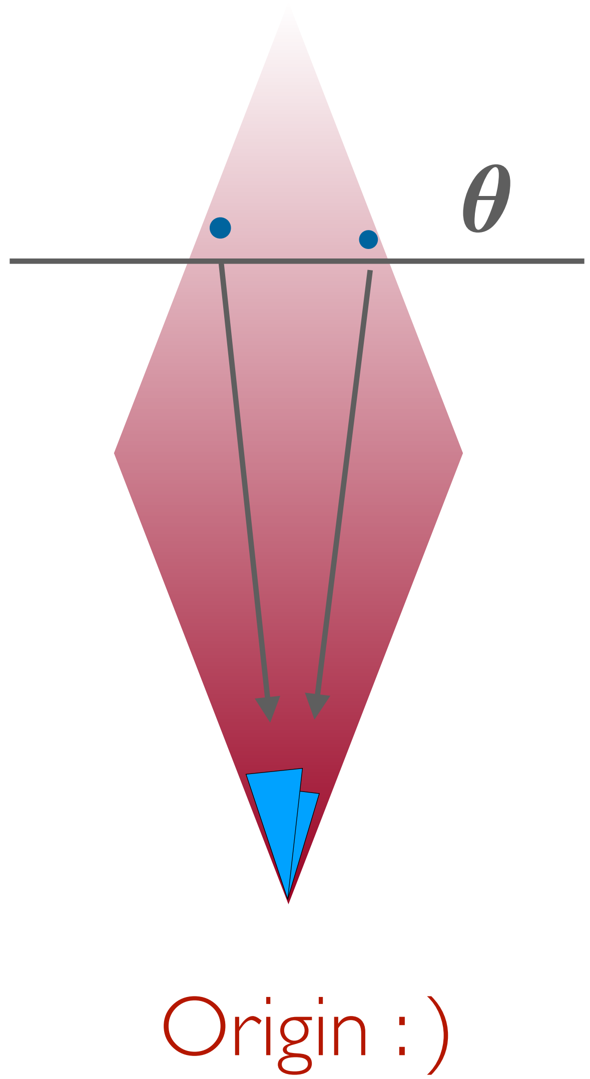

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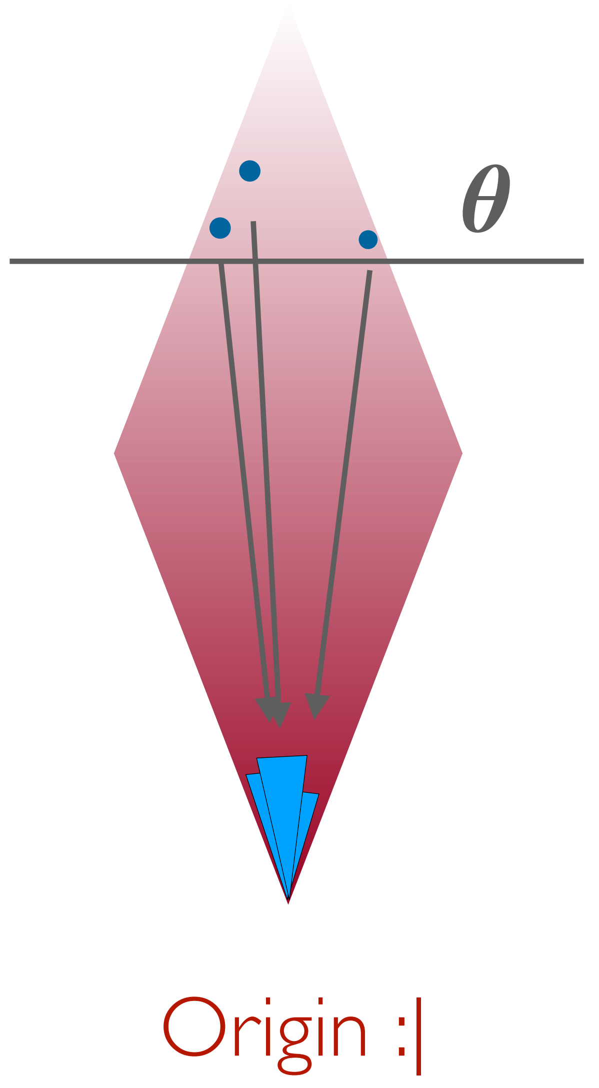

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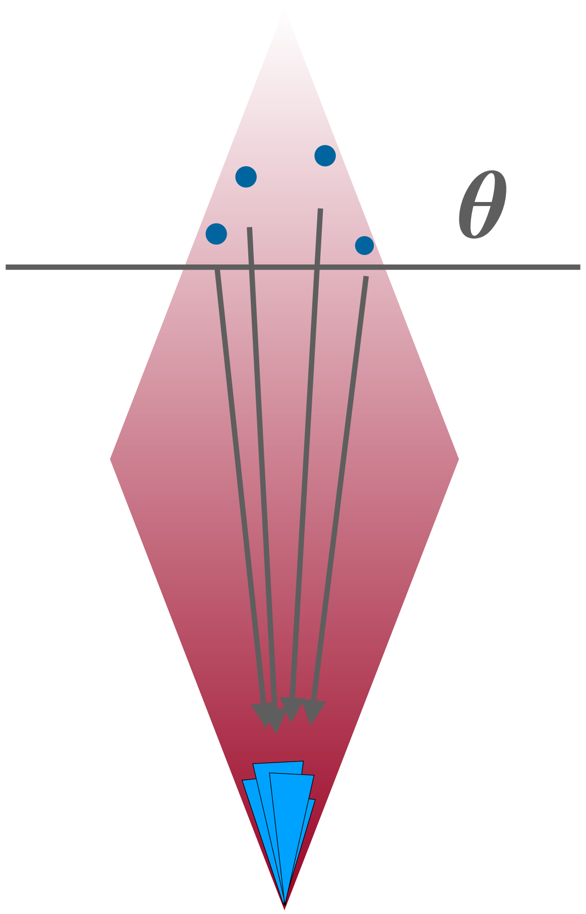
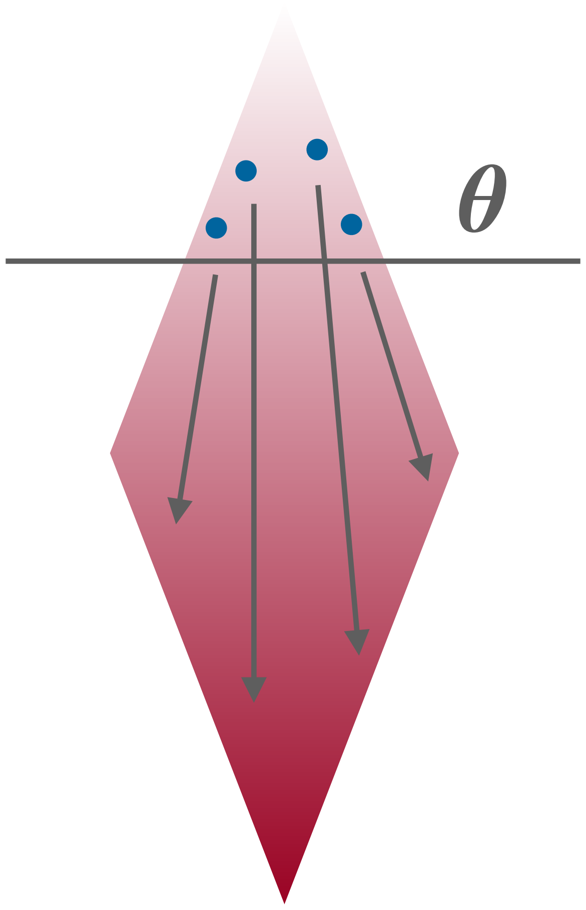


Overwhelms the origin!

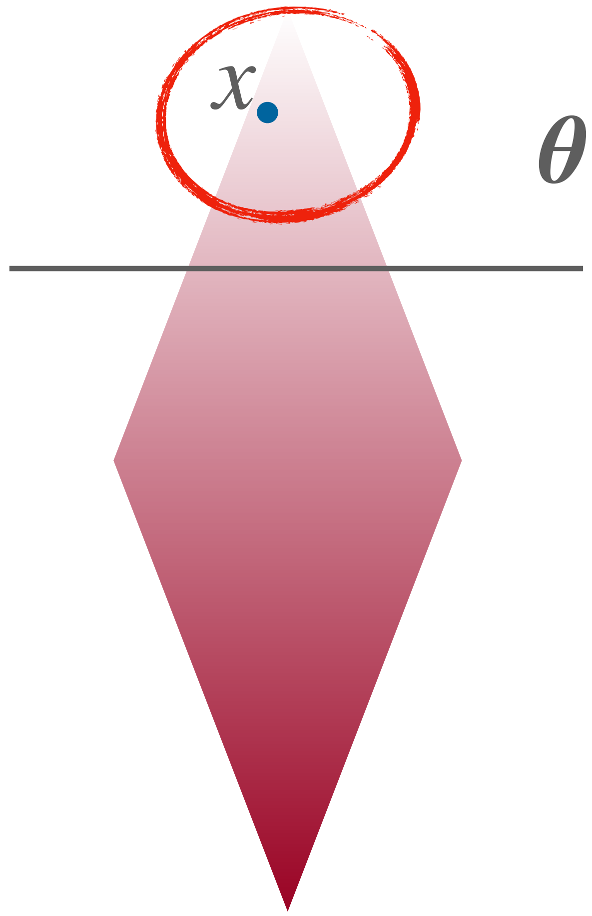
Origin : (

The idea



Spread the mass to different points

The idea



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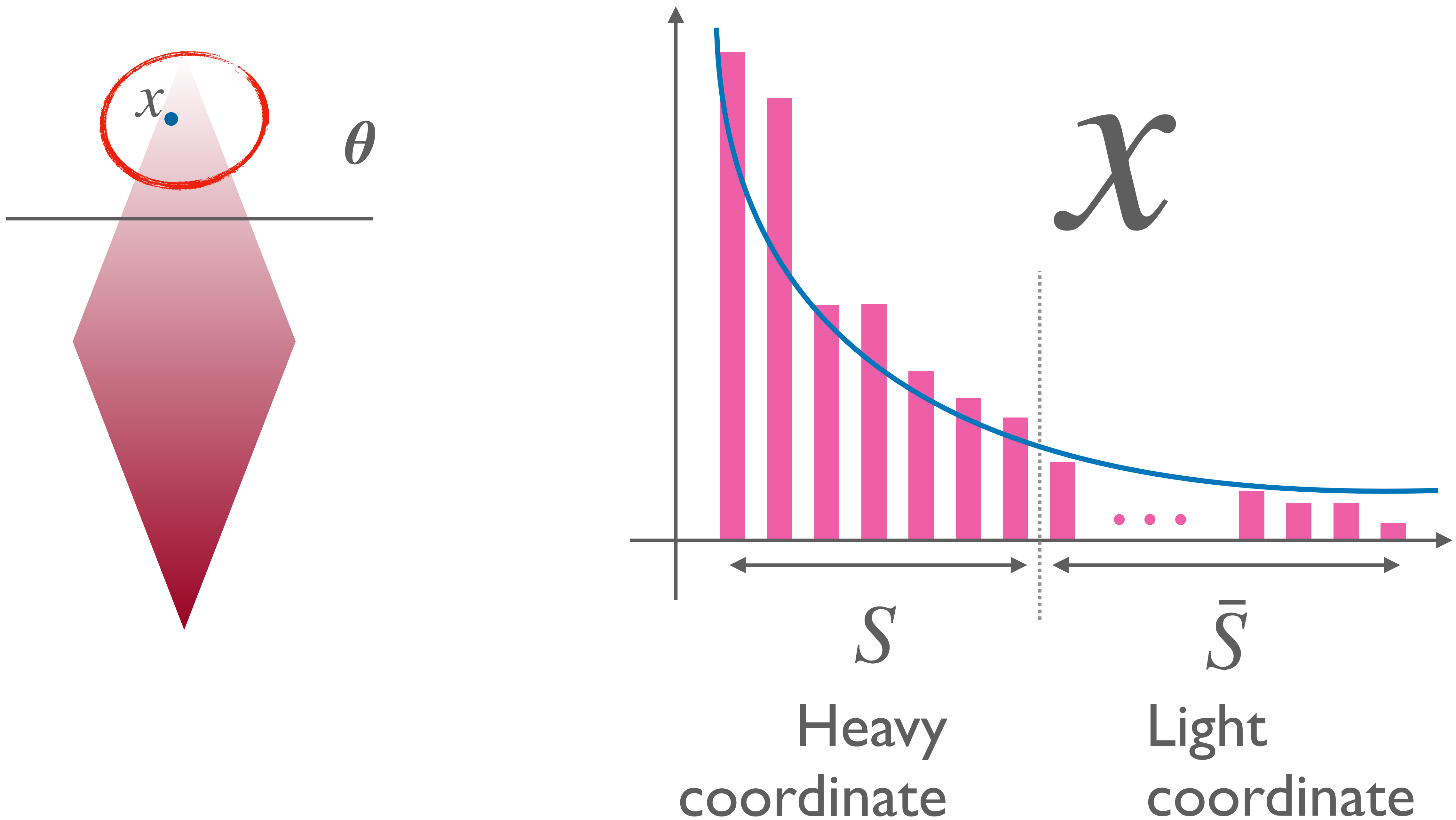

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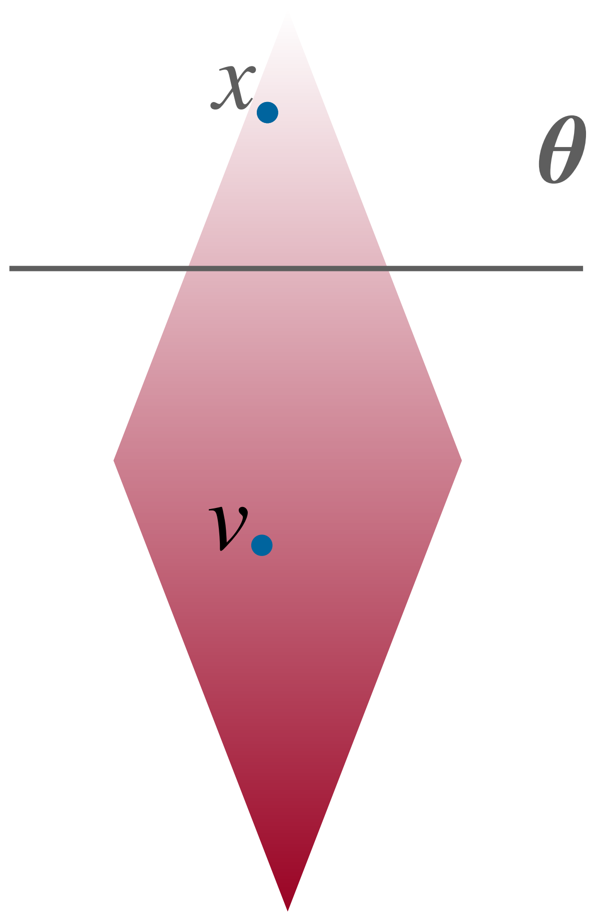

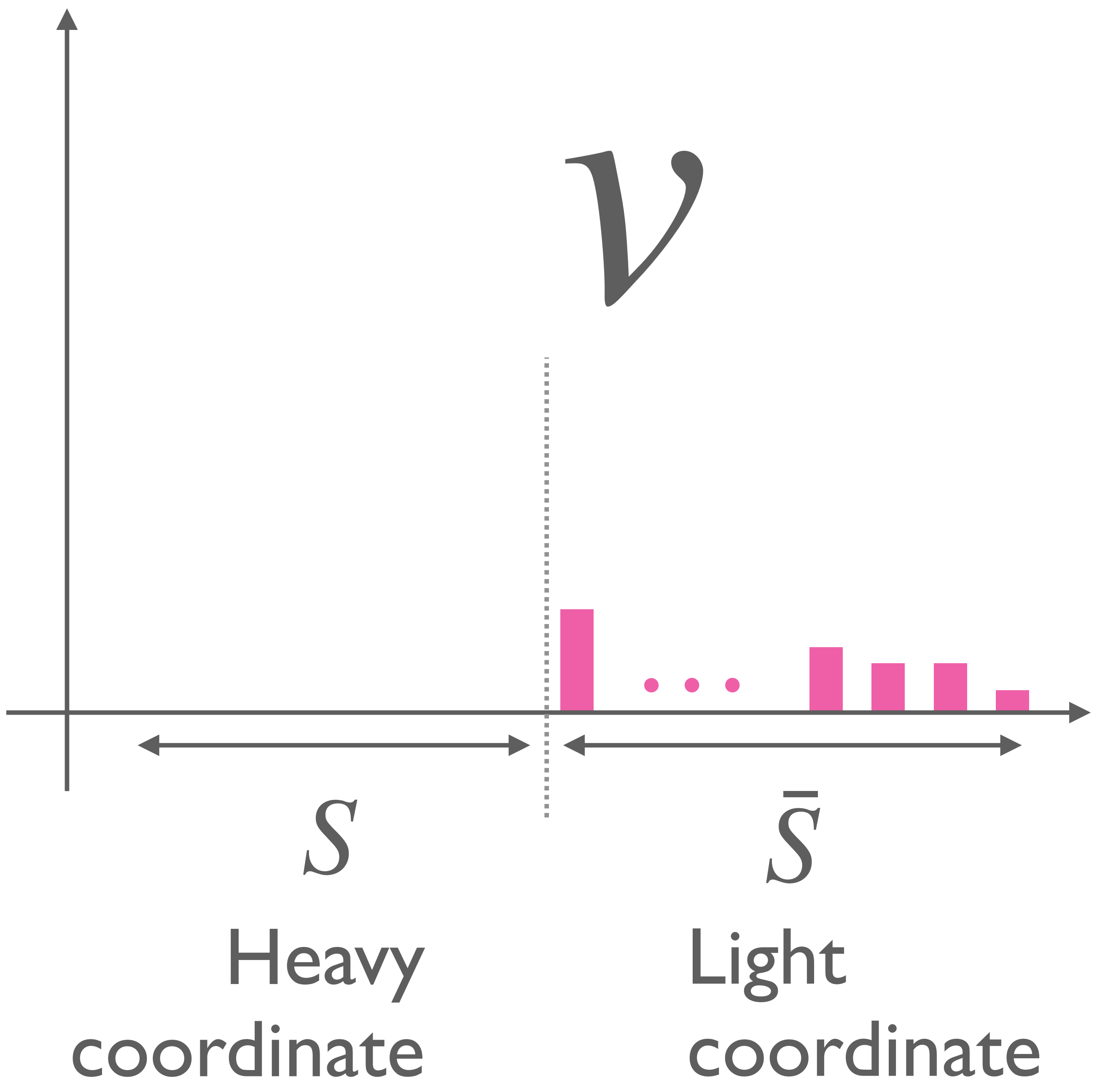

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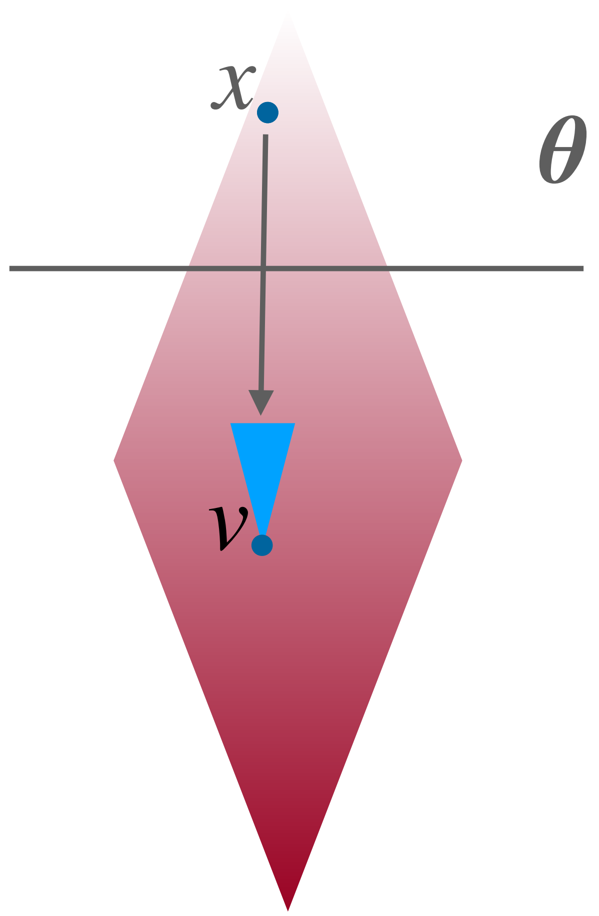

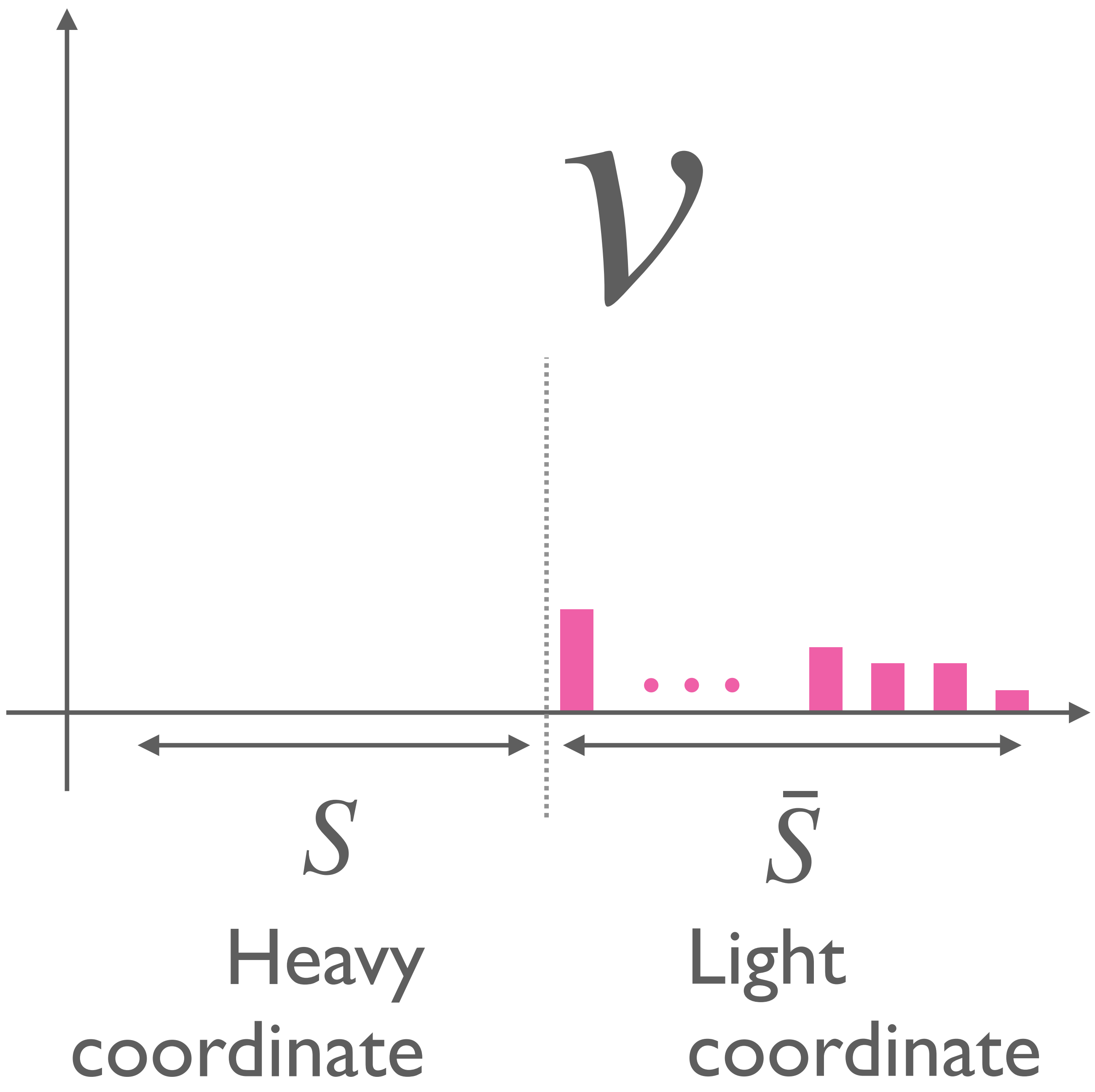

Part 2.
Sign Rank

Lift threshold degree to sign rank

threshold degree

$\downarrow$

Sign rank

Fact. ([Forster 01] + [Sherstov 08])

Given $\deg_{\pm}(f, 2^{-O(d)}) = d$,

Then, there is F that has $\mathrm{rk}_{\pm}(F) = \exp(\Omega(d))$.

Lift threshold degree to sign rank

“Smooth”
threshold degree

$\downarrow$

Sign rank

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Smooth threshold degree

$f : X \to \{0,1\}, \;\; X = \{0,1\}^n$

Definition.

$$\deg_{\pm}(f, \gamma) = \max\left\{ \operatorname{orth}(\mu \cdot (-1)^f) : \mu(x) \geq \frac{\gamma}{|X|} \right\}.$$

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γ-smooth dual object

Smooth threshold degree

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Definition.

$$\deg_\pm(f,\gamma) = \max\left\{ \operatorname{orth}(\mu \cdot (-1)^f) : \mu(x) \geq \frac{\gamma}{|X|} \right\}.$$

γ-smooth dual object

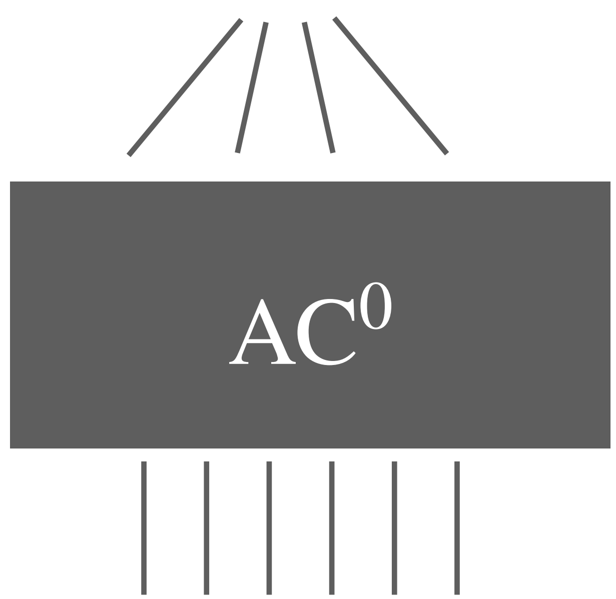
Fact. For any non constant $f : X \to \{0,1\}$,

$$\deg_\pm(f, 1/2) \geq 1.$$

Hardness ampl. of smooth threshold degree

Given $f : X \to \{0,1\}$, $\deg_{\pm}(f, \gamma) = n^{1-\epsilon}$.

Then $F = f$



$x, y \in \{0,1\}^N$

$$\deg_{\pm}(F, \gamma \exp(-\tilde{O}(N^{1-\frac{\epsilon}{1+\epsilon}})) = \tilde{\Omega}(N^{1-\frac{\epsilon}{1+\epsilon}}) .$$

Hardness ampl. of smooth threshold degree

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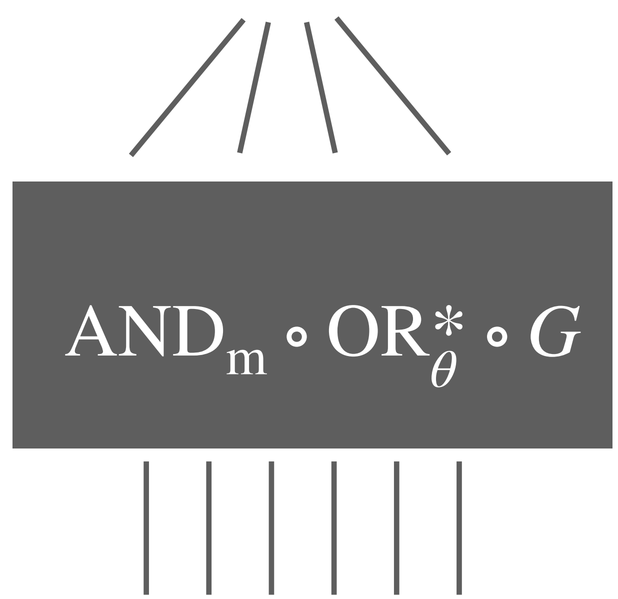

$x, y \in \{0,1\}^N$

Why we are not done?

Our dual is not for $\deg_{\pm}(f \circ \mathrm{AND_m} \circ \mathrm{OR}^*_\theta \circ G)$;

Highly non-smooth

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Highly non-smooth

dual of $f \circ \mathrm{MP}^*_m|_{\leq\theta}$

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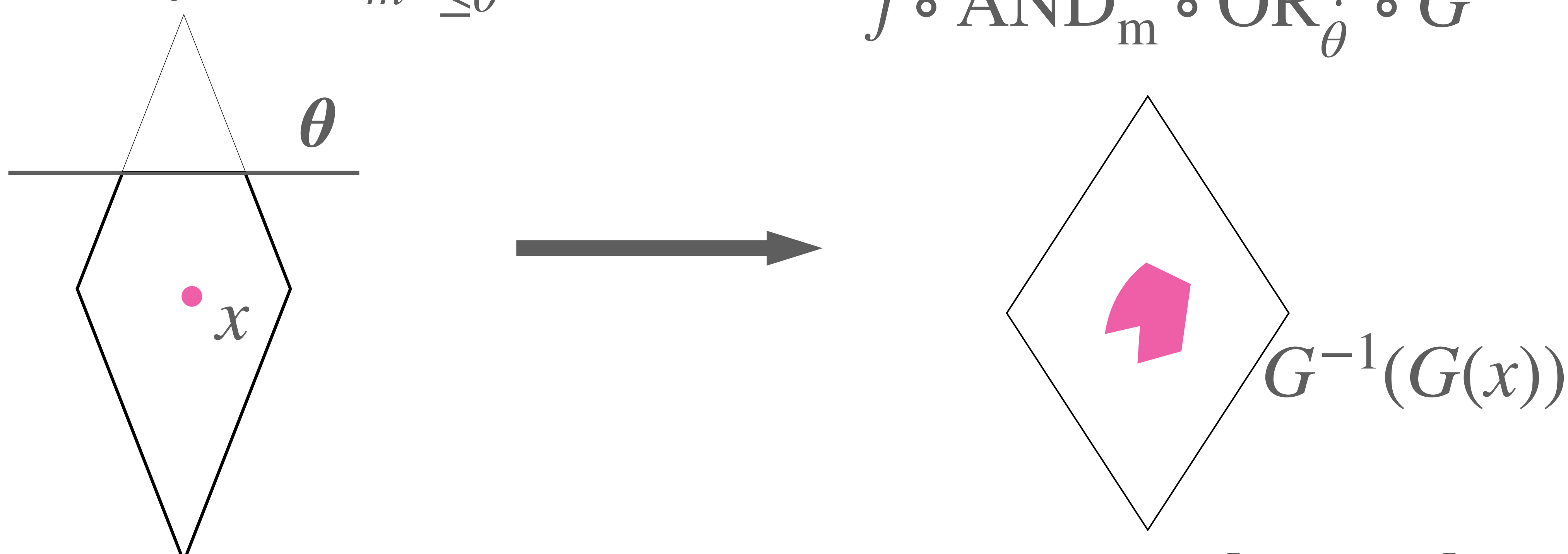

$\{0,1,\ldots,m^2\}^{nm}|_{\leq\theta}$

$\{0,1\}^{\theta\lceil 6\log(n+1)\rceil}$

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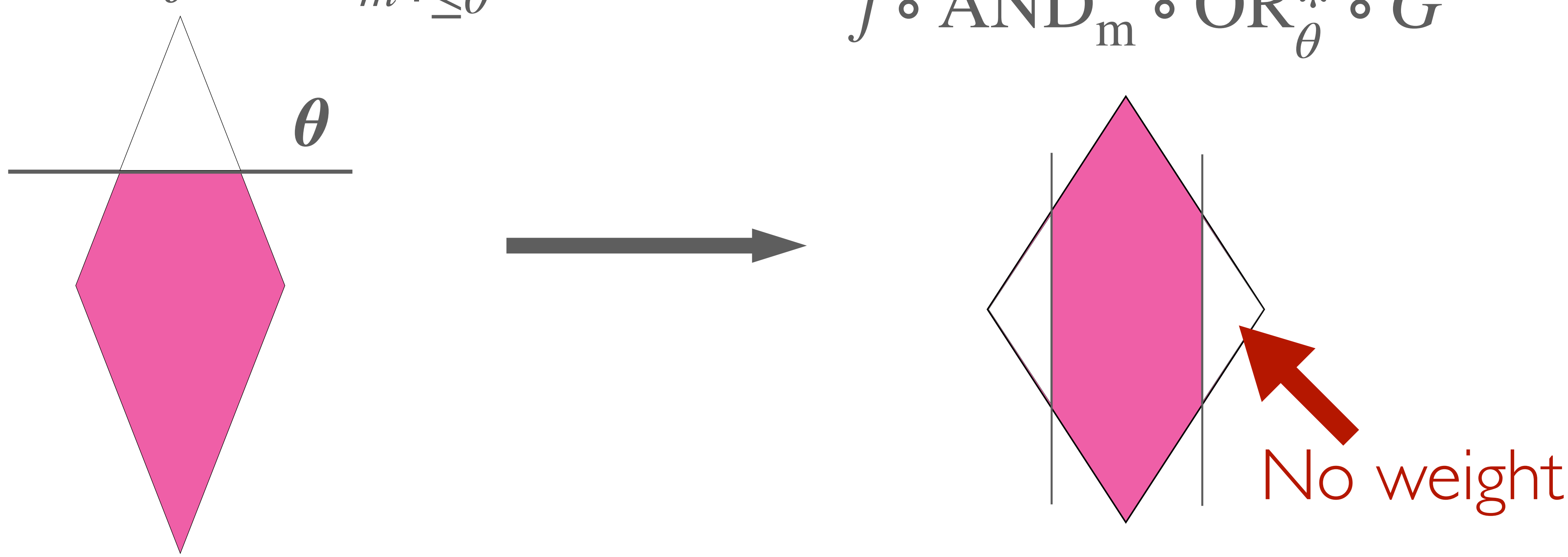

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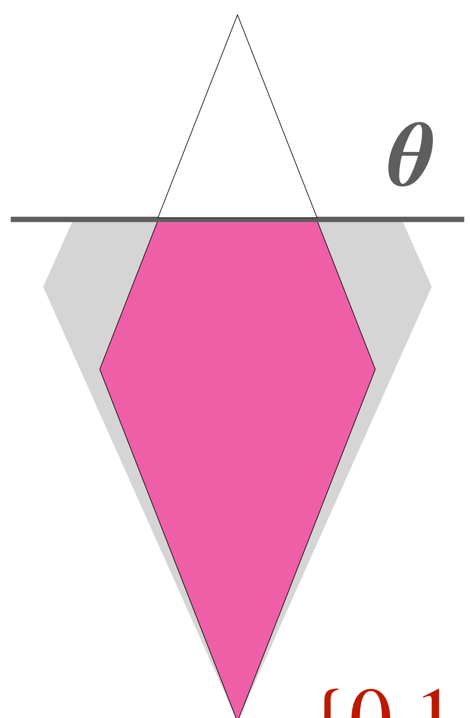


$\{0,1,\ldots,\theta\}^{nm}|_{\leq\theta}$

$\cancel{\{0,1,\ldots,m^2\}^{nm}}|_{\leq\theta}$

dual of

$f \circ \mathrm{AND_m} \circ \mathrm{OR}_{\theta}^{*} \circ G$

No weight on

1	0	0	0	0	0	0
1	0	0	0	0	0	0
1	0	0	0	0	0	0
1	0	0	0	0	0	0
1	0	0	0	0	0	0
1	0	0	0	0	0	0
1	0	0	0	0	0	0
1	0	0	0	0	0	0

Our approach: local smoothness

$\psi : X \to \mathbb{R}$,

Definition.

ψ is K-locally-smooth if

$$\left| \frac{\psi(x)}{\psi(y)} \right| \leq K^{\|x-y\|_1}$$

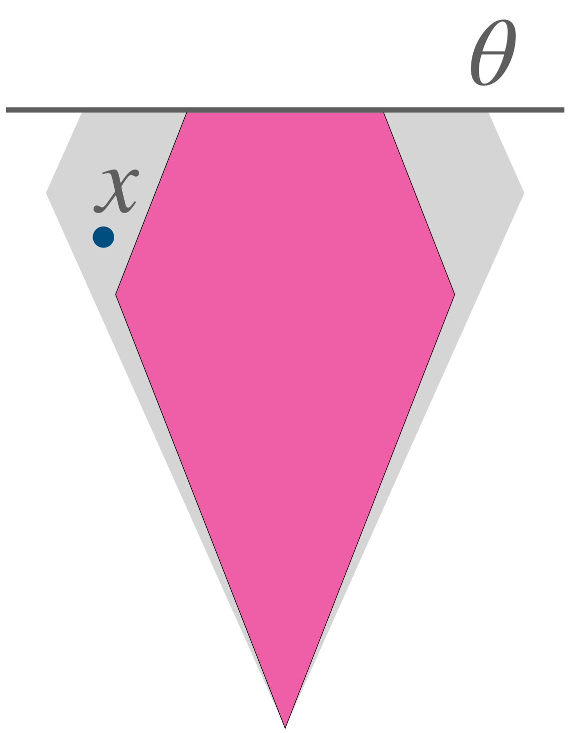
for any $x, y \in \mathrm{supp}(\psi)$.

Local smoothness is powerful

$\psi : X \to \mathbb{R}$, locally-smooth,

$X = \{0,1,\ldots,M\}^N|_{\leq\theta}$

$x \in \mathbb{N}^N|_{\leq\theta}$



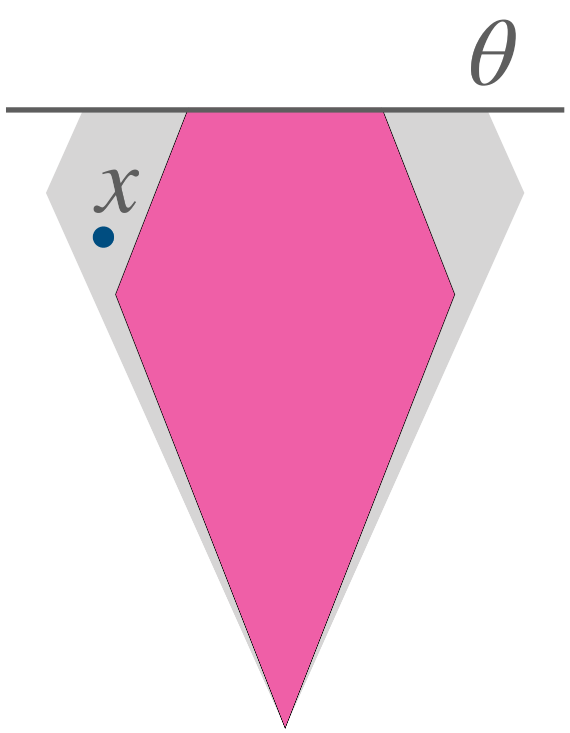
Local smoothness is powerful

A dual object of $f : \mathbb{N}^N|_{\leq\theta} \to \mathbb{R}$

$\psi : X \to \mathbb{R},$ locally-smooth,

$X = \{0,1,\ldots,M\}^N|_{\leq\theta}$

$x \in \mathbb{N}^N|_{\leq\theta}$



Local smoothness is powerful

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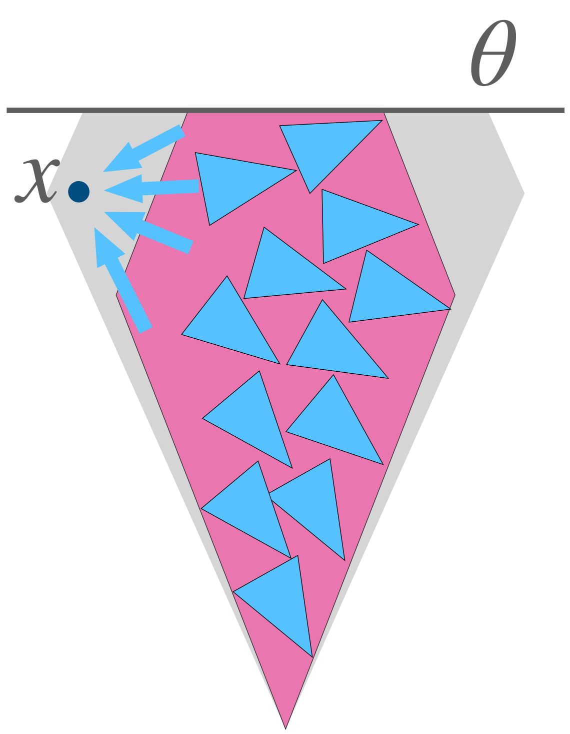


Pack X with balls of radius $O(d)$

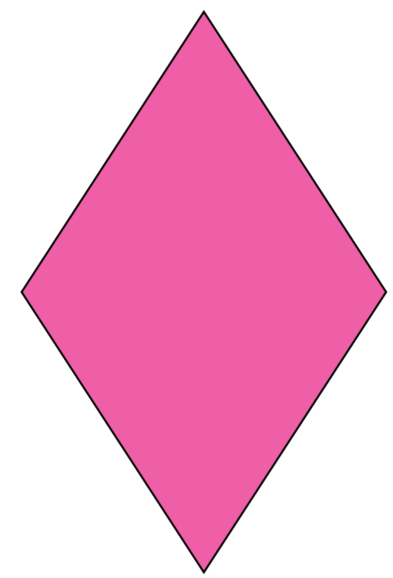
$\mathscr{B} =$ collection of "correctors" ζ_v labeled by the lightest point.

$$\tilde{\psi}_x = \psi + (-1)^{f(x)} \sum_{\zeta_v \in \mathscr{B}} \frac{|\psi(v)|}{\|\zeta_v\|_1} \zeta_v .$$

The ideal dual object

ideal dual of

$f \circ \mathrm{AND_m} \circ \mathrm{OR}^*_\theta \circ G$



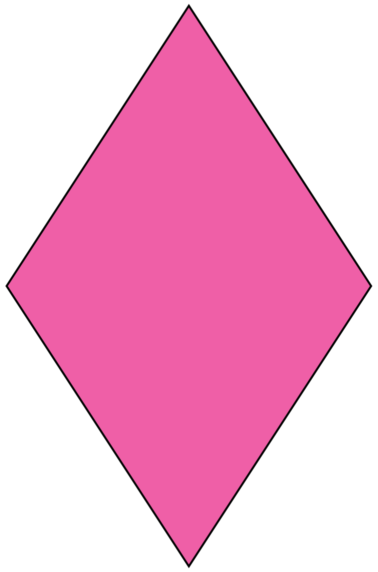
$\{0,1\}^{\theta\lceil 6\log(n+1)\rceil}$

uniform

The ideal dual object

ideal dual of

$$f \circ \mathrm{AND_m} \circ \mathrm{OR}^*_\theta \circ G$$



$$\{0,1\}^{\theta\lceil 6\log(n+1)\rceil}$$

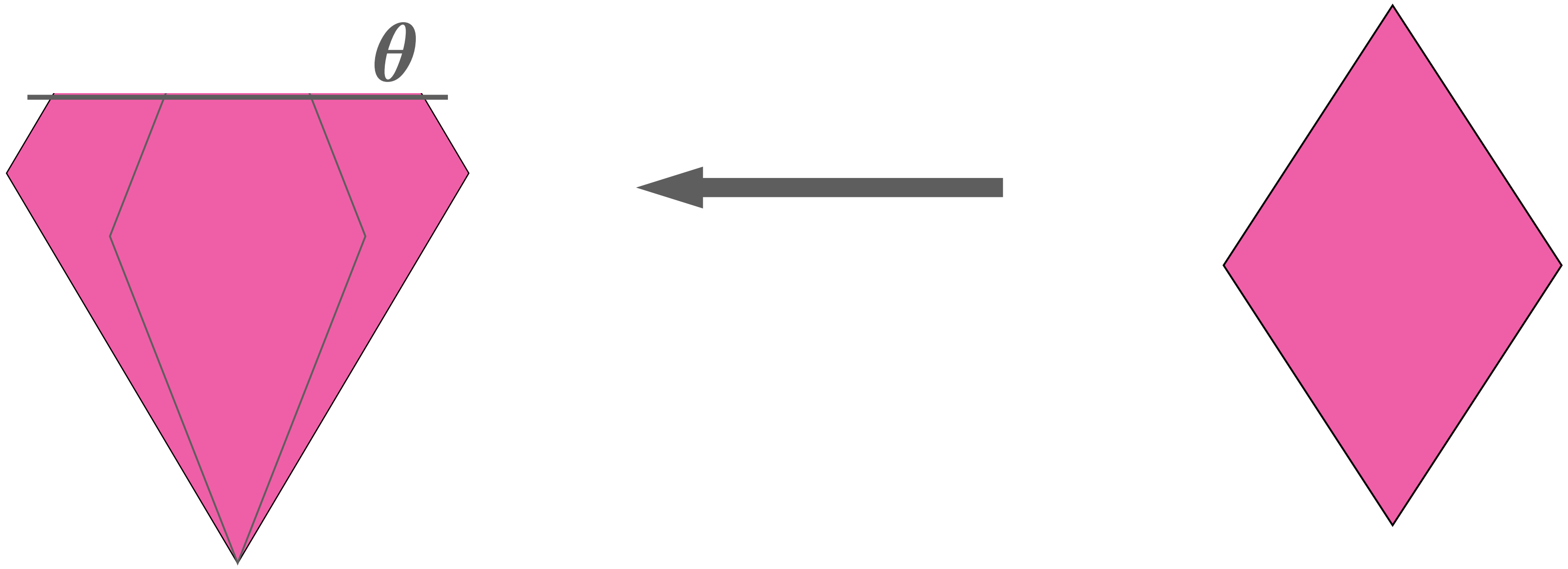
uniform

unrealistic

The ideal dual object

ideal dual of
$f \circ \mathrm{AND}_{\mathrm{m}} \circ \mathrm{OR}^*_{\theta}$

ideal dual of
$f \circ \mathrm{AND}_{\mathrm{m}} \circ \mathrm{OR}^*_{\theta} \circ G$



$\mathbb{N}^{nm}|_{\leq\theta}$

Λ^*

$\{0,1\}^{\theta\lceil 6\log(n+1)\rceil}$

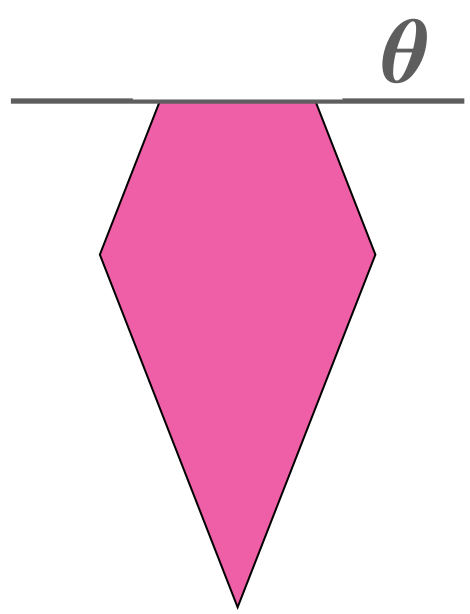
uniform

unrealistic

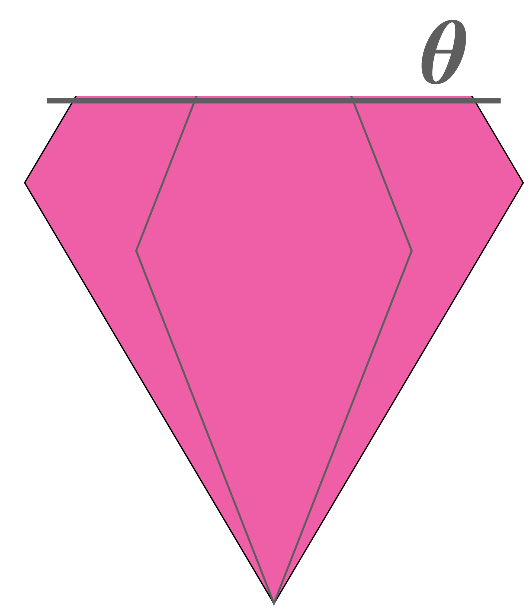
Toward the ideal dual object

dual of $f \circ \mathrm{MP}^*_m|_{\leq\theta}$

ideal dual of
$f \circ \mathrm{AND_m} \circ \mathrm{OR}^*_\theta$



Shift weight



$\{0,1,\ldots,m^2\}^{nm}|_{\leq\theta}$

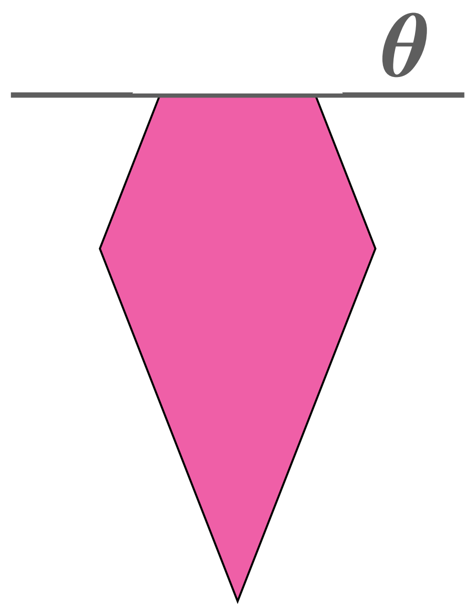
$\mathbf{\Psi}$

$\mathbb{N}^{nm}|_{\leq\theta}$

Λ^*

Toward the ideal dual object

dual of $f \circ \mathrm{MP}^*_m|_{\leq\theta}$



$\{0,1,\ldots,m^2\}^{nm}|_{\leq\theta}$

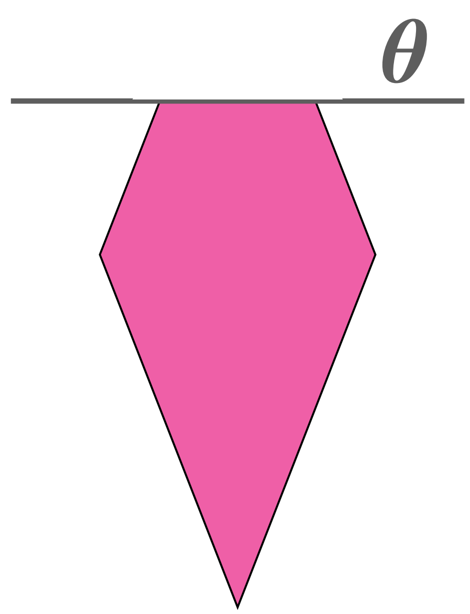
Ψ

ψ γ-smooth dual object of f

$$\begin{aligned}\Psi &= \sum \psi(z)\tilde{\Lambda}_z \\ &= \sum (\psi(z) - \gamma(-1)^{f(z)})\tilde{\Lambda}_z \\ &\qquad +\gamma \sum (-1)^{f(z)}\tilde{\Lambda}_z\end{aligned}$$

Toward the ideal dual object

dual of $f \circ \mathrm{MP}^*_m|_{\leq\theta}$



$\{0,1,\ldots,m^2\}^{nm}|_{\leq\theta}$

Ψ

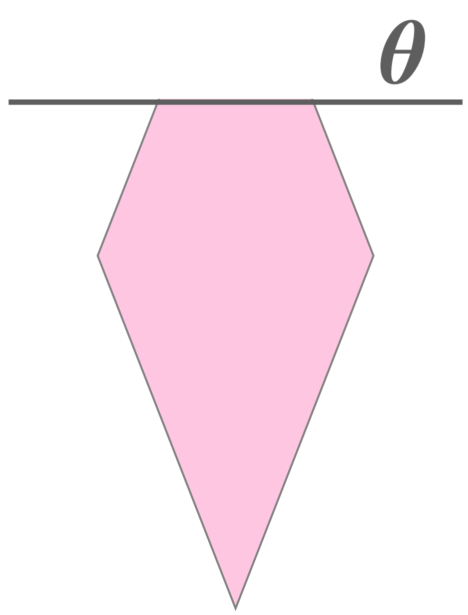
ψ γ-smooth dual object of f

$$\Psi = \sum \psi(z)\tilde{\Lambda}_z$$
$$= \sum (\psi(z) - \gamma(-1)^{f(z)})\tilde{\Lambda}_z$$
$$+\gamma \sum (-1)^{f(z)}\tilde{\Lambda}_z$$

Λ

locally smooth

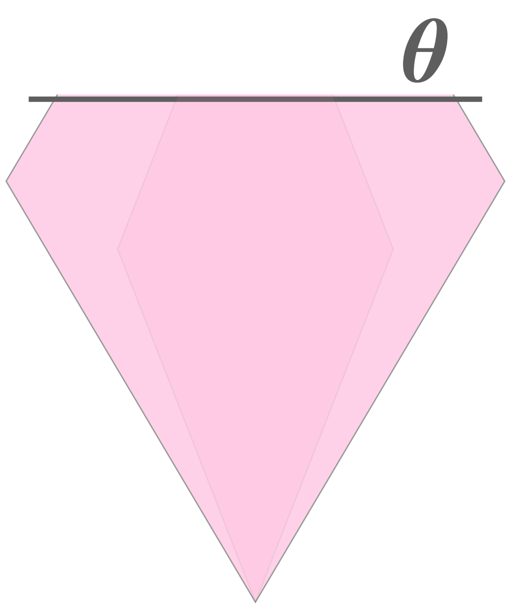
Toward the ideal dual object



$\gamma\Lambda$

1. Apply Theorem 2 at each point in $\mathbb{N}^{nm}|_{\leq\theta}$,

2. Take the convex combination



$\gamma\tilde{\Lambda}^*$

$$\tilde{\Lambda}^* = \sum_{x \in \mathbb{N}^{nm}|_{\leq\theta}} \Lambda^*(x) \cdot \tilde{\Lambda}_x$$

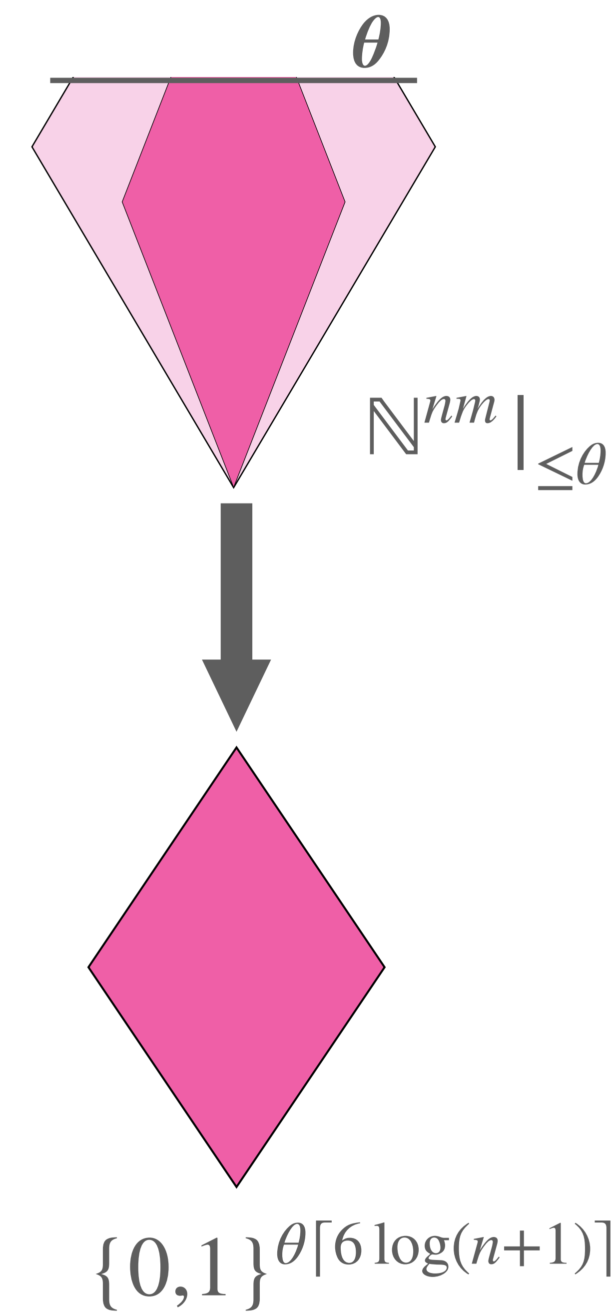
$$\begin{aligned}
&\mathrm{orth}(\tilde{\Lambda}^* - \Lambda) \geq d, \\
&\tilde{\Lambda}^* \cdot (-1)^F \geq 0, \\
&|\tilde{\Lambda}^*| \geq (nmK)^{-O(d)}|\Lambda^*|, \\
&\|\tilde{\Lambda}^*\|_1 \leq 2\|\Lambda\|_1.
\end{aligned}$$

Finishing the proof on hardness ampl.

$\Psi = \mathit{whatever} + \gamma\Lambda,$ dual of $f \circ \mathrm{MP}^*_m|_{\leq\theta}$

1. Construct $\gamma(nmK)^{-O(d)}$-smooth dual of $f \circ \mathrm{AND_m} \circ \mathrm{OR}^*_\theta$ w.r.t. Λ^*

$$\tilde{\Psi} = \mathit{whatever} + \gamma \sum_{x \in \mathbb{N}^{nm}|_{\leq\theta}} \Lambda^*(x) \cdot \tilde{\Lambda}_x .$$



Finishing the proof on hardness ampl.

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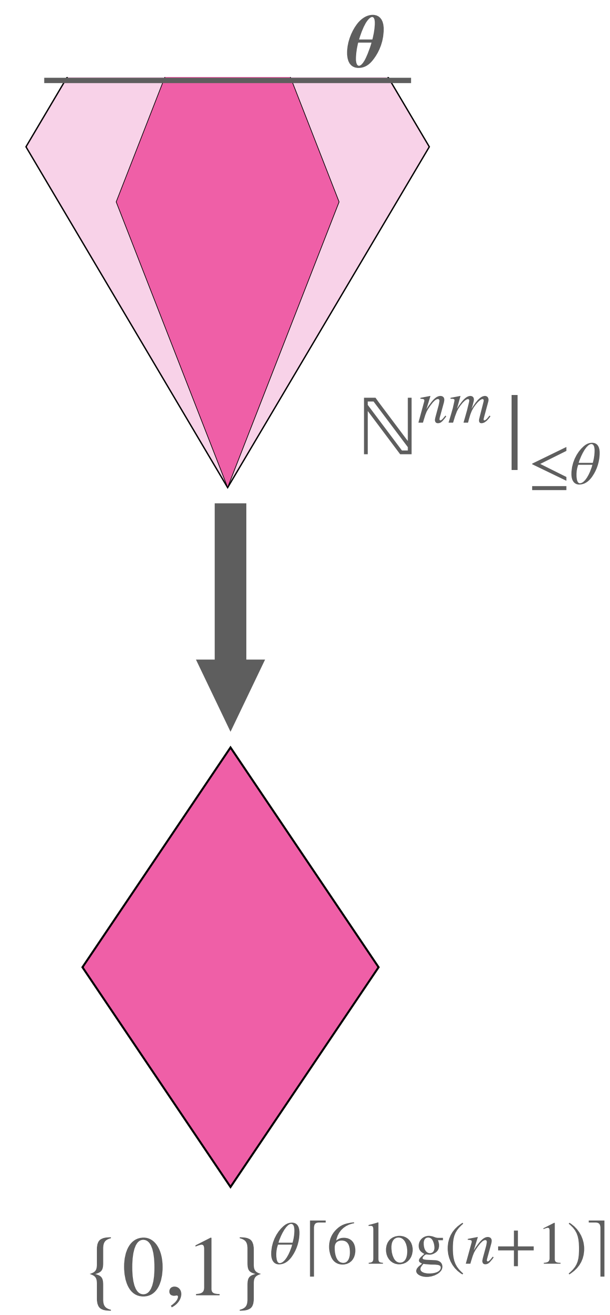
$$\tilde{\Psi} = whatever + \gamma \sum_{x \in \mathbb{N}^{nm}|_{\leq\theta}} \Lambda^*(x) \cdot \tilde{\Lambda}_x .$$

2. Convert $\tilde{\Psi}$ to a $\gamma(nmK)^{-O(d)}$-smooth dual of $f \circ \mathrm{AND_m} \circ \mathrm{OR}^*_\theta \circ G$.



Finishing the proof on hardness ampl.

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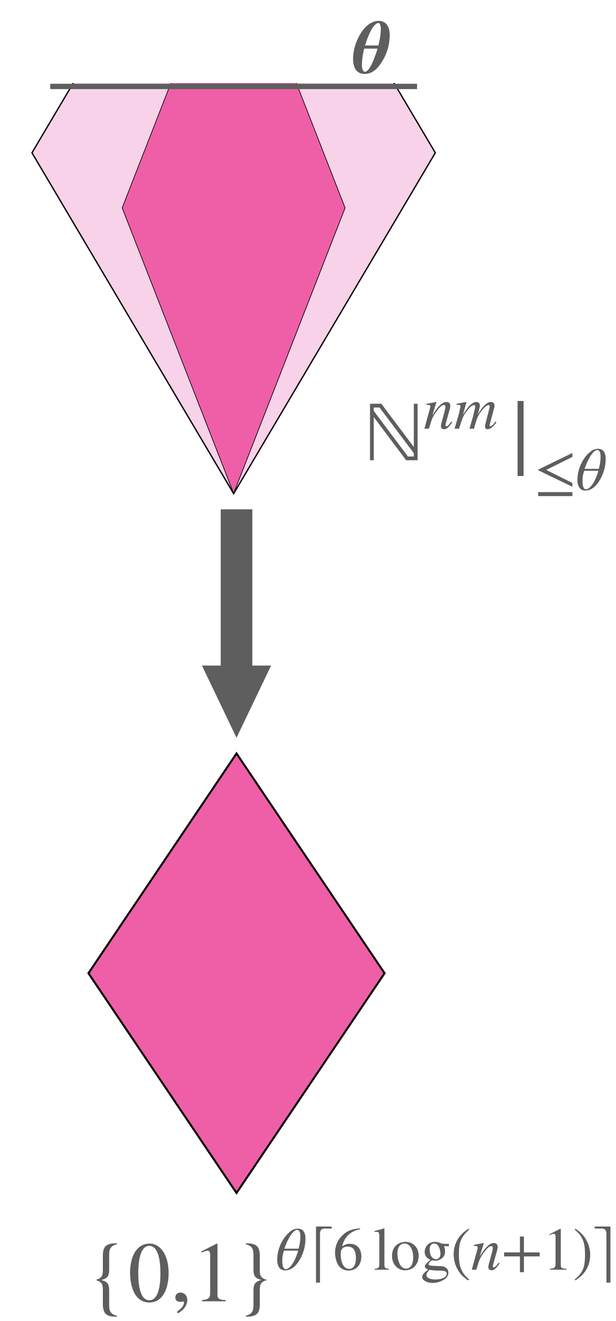
1. Construct $\gamma(nmK)^{-O(d)}$-smooth dual of $f \circ \mathrm{AND_m} \circ \mathrm{OR}^*_\theta$ w.r.t. Λ^*

$$\tilde{\Psi} = whatever + \gamma \sum_{x \in \mathbb{N}^{nm}|_{\leq\theta}} \Lambda^*(x) \cdot \tilde{\Lambda}_x \, .$$

2. Convert $\tilde{\Psi}$ to a $\gamma(nmK)^{-O(d)}$-smooth dual of $f \circ \mathrm{AND_m} \circ \mathrm{OR}^*_\theta \circ G$.

3.* Construct locally-smooth dual object of MP^*_m .



Open problems

Problem 1.

$$\deg_{\pm}(\mathrm{AC}^0) \geq \frac{n}{2020}?$$

Problem 2.

$$\mathrm{rk}_{\pm}(\mathrm{AC}^0) \geq \exp\left(\frac{n}{2020}\right)?$$

Problem 3.

$$\deg_{1/3}(\mathrm{AC}^0) \geq \frac{n}{2020}?$$

Thank you!